Foundations of Corporate Finance

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Contents

Preface v

Chapter 1. An Overview of the Financial System & Corporate Finance 1
  1. An Overview of the Financial System 1
  2. Financial Markets 2
  3. Financial Instruments 7
  4. Financial Intermediaries & Institutions 9
  5. The Corporate Environment 11
  6. Practice Problems 16

Chapter 2. Risk and Return 17
  1. Preliminaries: Returns as Random Variables 17
  2. Sample Mean, Population Mean, and the Rule of Expectations 21
  3. Return: The Meaning of the Expected Rate of Return 27
  4. The Variance of a Random Variable 28
  5. Risk: The Meaning of the Standard Deviation 33
  6. Covariance 37
  7. Correlation 41
  8. Financial Application: Constructing a Portfolio 44
  9. Attitudes Towards Risk 47
  10. Practice Problems 50

Chapter 3. Capital Asset Pricing Model 53
  1. The Axioms of CAPM 55
  2. The Rationale of CAPM 57
  3. The Security Market Line (SML) 65
  4. CAPM in Practice 70
  5. Practice Problems 75

Chapter 4. Time Value of Money 79
  1. Future Values 79
  2. Present Values 81
  3. PV and FV of Multiple Cash Flows 84
  4. Perpetuities and Annuities 85
  5. Effective Annual Interest Rates 91
  6. Practice Problems 94

Chapter 5. Bond and Stock Valuation 97

1. Asset Valuation: A Little Bit of History 97
2. Bond Valuation 102
3. Cost of Debt & Yield to Maturity 105
4. Interest Rate and Bond Prices 111
5. Some Practice Problems on Bond Valuation 113
6. Stock Valuation 117
7. The Dividend Discount Model 120
8. Valuing Preferred Stocks 128
9. Some Practice Problems on Stock Valuation 128
10. Practice Problems 131

Chapter 6. Capital Budgeting 135
1. The Weighted Average Cost of Capital 136
2. Initial Investment 141
3. Operating Cash Flow 146
4. Terminal Cash Flow 148
5. Practice Problems 151

Chapter 7. Budgeting Techniques 155
1. Payback Period 155
2. DCF Methods: NPV & IRR 156
3. Practice Problems 159

Chapter 8. Financial Evaluation in Practice 163
1. Financial Evaluation: Sources of Information 163
2. Financial Evaluation: The Methodology 172
3. Step 1: Analyzing Revenues and Costs 172
4. Step 2: Capital Required to Finance a Project 173
5. Step 3: Proper Financial Structure 176
6. Step 4: Financial Evaluation 179
7. Step 5: Sensitivity Analysis 184
8. Practice Problems 186

Chapter 9. Capital Structure and Dividend Policy 189
1. Introduction: Financial Leverage 189
2. The Theory of Capital Structure 192
3. Dividend Policy 198

Bibliography 205

Preface

This textbook is intended as a one-term text for an introductory course in corporate finance at the senior or the first-year graduate level. The book follows a hands-on approach that matches the theory with practice. It is written for practitioners and non-finance major students who are interested to learn or pursue a career in finance. Although an exposure to an introductory course in statistics and linear algebra is always preferable, it is not mandatory to read the content of this textbook and easily follow its arguments. The reader, however, is expected to bring the background of an elementary course in algebra. My intention from writing this textbook is to provide the readers a concise and rigorous introductory course in corporate finance that is a balanced blend between the theory and its practical applications.

This book is organized as follows: Chapter 1 is an overview of the financial system and the corporate environment. Chapter 2 discusses what risk and return mean in finance. It explains how investors and financial analysts compute risk and return on individual assets and portfolio of assets in practice. This chapter is considered an introduction to finance theory, which is a branch of finance that is after developing equilibrium models of price formation. In Chapter 3, we discuss thoroughly one equilibrium model of price formation; the so-called capital asset pricing model. Together Chapters 2 and 3 constitute the foundations to finance theory. The remaining part of the textbook is devoted to financial valuations. Chapter 4 is about time value of money, which is an introduction to valuations. In Chapter 5, the reader will learn how to value bonds and stocks by applying the conventional discounted cash flow formula to financial instruments. In Chapters 6 and 7, the reader will learn how to value corporate assets. Chapter 8 is considered a practical financial application on valuation. In particular, in this chapter, we discuss how to upraise simple financial projects. Finally, Chapter 9 explores the theory of capital structure and dividend policy.

CHAPTER 1

An Overview of the Financial System & Corporate Finance

In this chapter we give an introduction to the notion of **finance** and the **financial system** in general. The intention from this introductory chapter is to give the reader a sense of the financial environment and to identify the main instruments and players in financial markets.

1. An Overview of the Financial System

Before we explore the financial system, it is, perhaps, constructive to take a quick look at the structure of any economy from a macro economic perspective. After all, finance is considered a branch of economics. Any society/economy, from a macro perspective, consists of four sectors, namely, a **household** sector, a **business** sector, a central authority, i.e., a **government**, and a **foreign** sector. The household sector consists of all **consumers** and the business sector consists of all **producers/firms** in the economy. Consumers allocate their income between consumption and saving. Acting as lenders, consumers can either save their funds directly by engaging in a financial transaction, i.e., buying corporate bonds or stocks in the financial markets, or indirectly by simply depositing their saving into a saving bank account. If the saving is done directly, then the money is channelled from the savers/consumers/lenders to corporations/producers/borrowers directly through trading corporate bonds and stocks in the financial market. If, on the other hand, the saving is done indirectly, then the funds is channeled from lenders to borrowers through a financial intermediary, i.e., the bank, which acts as a middleman between lenders and borrowers.

Definition 1. *Finance is the study of why and how saved funds are allocated between lenders and borrowers.*

At this point, to understand finance as a concept, we need to answer two questions: First, why households or consumers save? Second, how the amount allocated for saving by households is actually saved, i.e., how savings are channelled in the financial system? The answers to these questions explain how the financial system works.

To answer the first question, one need to understand the consumers' motives to allocate their income on consumption and saving. Economic theory tells us that consumers tend to consume in order to maximize their satisfaction from consumption of goods and services. They tend to save in order to **smooth** their consumption over time.

Definition 2. *Consumption smoothing means setting aside, or saving, some income today in hope of achieving some return on it in the future in order to maintain the same standard of living.*

The idea here is that, over the life cycle of any individual, the process of generating income is not the same as consumption spending; income is received at discrete dates (weekly or monthly) while consumption is, in a way, continuous (daily). Since individuals like to maintain the same standard of living over their life cycle, they prefer smoothing their consumption over time. This answers why consumers save. Now, we turn to how the saving is done.

Corporations are producers or businesses who raise funds to invest in order to increase their profits in the future. They usually finance their investment expenditures through **borrowing** either directly by issuing bonds or stocks in the financial market or indirectly by requesting loans from a financial intermediary, e.g., a commercial bank. Thus, **corporations/producers** as economic agents are technically the **borrowers** in the financial system depicted in Figure 1. But, what about the lenders? Who are the economic agents that are considered lenders in the financial system? Lenders in the financial system are individuals or institutions who have saved some money and want to achieve a return on their savings in order to smooth their consumption. Thus, the **lenders** in the financial system are the **savers/consumers** in the economy. Funds are channeled from lenders to borrowers either directly through buying and selling bonds and stocks in the financial market or indirectly by means of financial intermediaries as shown from the flow of funds in the financial system depicted in Figure 1. In the following two sections, we explore financial markets and financial intermediaries in more details.

2. Financial Markets

Definition 3. *A market is a place where buyers and sellers of a good or a service meet to undertake transactions. The market participants are the buyers and the sellers of the good or service under consideration. The buyers constitute the demand side of the market and the sellers constitute the supply side.*

There are many markets in any economy, e.g., markets for goods and services, markets for money, markets for bonds and stocks, labor market, and

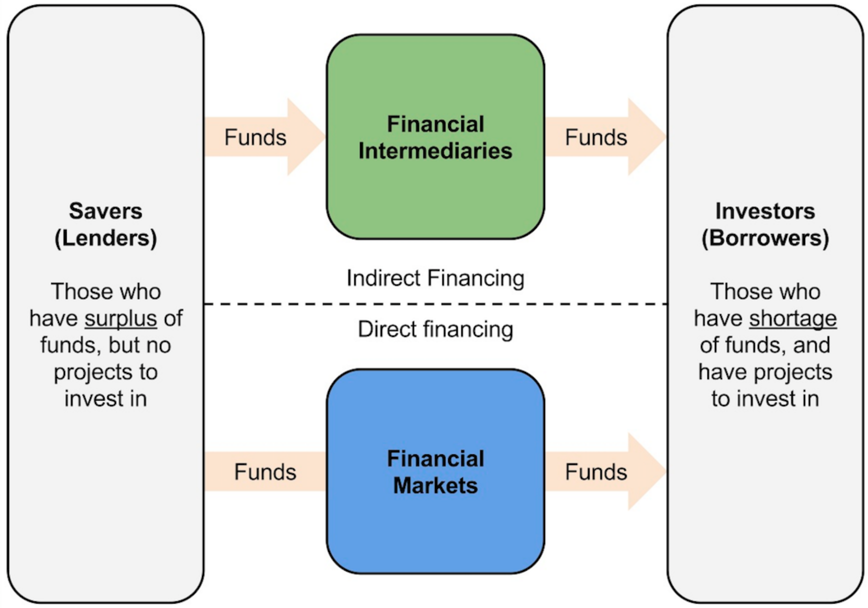

Figure 1. The Financial System

so on. We shall confine ourselves to financial markets. These markets are the part of the financial system where funds are transferred from lenders to borrowers. In what follows, we look at the lenders and borrowers, the channels of intermediations, and the instruments and institutional arrangements that are used to transfer funds in these markets.

Definition 4. *Financial instruments are financial assets taking the form of legal documents that define the rights and obligations of all the parties involved, i.e., the borrowers and the lenders. Financial instruments, also known as* ***financial securities****, are broadly classified into two main groups:* ***debt*** *instruments and* ***equity*** *instruments.*

2.1. Classifications of Financial Markets.

Financial markets can be grouped into different classifications according to their characteristics. For instance, they can be classified according to product type, liquidity, time period, size, geographical focus, trading schedule, and many other attributes. Four conventional classifications of financial markets usually appear in most finance and economic textbooks, namely: (1) primary and secondary markets, (2) debt and equity markets, (3) exchange and over-the-counter (OTC) markets, and (4) money and capital markets. Below is a brief description of each category.

2.1.1. *Primary & Secondary Markets.* Markets where financial securities are issued for the first time are called **primary markets**. The first attempt by a business to issue ownership shares to the public in the primary market is called **initial public offering** or IPO for short. The function of primary markets, i.e., first issuance, is facilitated by investment banks that act as **underwriters**.

In primary markets, securities are bought and sold only at their face value. Markets where securities are resold by investors after they have already been sold on the primary market are called **secondary markets**. In secondary markets, already issued securities are traded at their market values, which are different than the face values for they are determined based on the market demand and supply forces. Most of traditional investing activities take place in secondary markets. The function of a secondary market, i.e., trading securities, is facilitated by **financial brokers**, who assist in matching borrowers and lenders of securities and earn a fee.

2.1.2. *Debt & Equity Markets.* Financial markets can also be classified according to the type of products traded into debt and equity markets. **Debt markets** are markets where debt instruments, e.g., bonds, are traded. **Equity markets** are markets where equity instruments, e.g., stocks are traded.

A debt instrument is a legal obligation by which the issuer/borrower is obliged to repay the holder/lender the principal amount borrowed plus interim interest payments at a specific maturity date. Maturity of a debt instrument is the time of that instrument's expiration date. Short-term debt instruments have maturity less than a year. Intermediate-term debt instruments have maturity between one and ten years. Long-term debt instruments have maturity of 10 years or longer. Common examples of debt instruments include bank loans, corporate bonds, treasury bills, bankers' acceptances, and mortgage loans.

An equity instrument, on the other hand, is an ownership stake in a corporation. There are two common forms of equity instruments: common shares and preferred shares. A common share is an equity instrument that represents ownership in a corporation and gives voting rights on major corporate decisions. Common shares are claims to share in net income and assets of a business. For instance, if a company issues 1 million shares and you own 1 share, then you are entitled to 1/million of the firm's net income and 1/million of the firm's assets. Common shares give periodic payments to their holders in the form of dividends. Common shares are considered long term securities since they don't have maturity dates. A preferred share is an equity instrument by which the owner is entitled a fixed dividend payments for ever. The preferred share dividends must be made before any dividends are paid to common shareholders. For this reason, preferred shareholders are sometimes known as first claimants.

2.1.3. *Exchange & OTC Markets.* A third classification of financial markets is according to the way they are organized. **Exchange markets** are markets where buyers and sellers, or their brokers, meet together in one central location to trade, i.e., buy and sell, securities at market prices. Exchange markets are transparent and regulated. **Over-the-counter (OTC) markets**, on the other hand, are considered decentralized markets, where dealers quote prices that they are willing to buy and sell at. Thus, in an OTC market, prices are not determined by demand and supply forces. These types of markets are less transparent and less regulated as opposed to exchange markets.

2.1.4. *Money & Capital Markets.* Finally, financial markets can be classified in terms of liquidity into **money markets** and **capital markets**. Money markets are markets where short-term debt instruments, with maturity less than one year, are traded. For example, three and six-month treasury bills are money market.

Capital markets are markets where long-term debt and equity instruments, with maturity more than one year, are traded. It is worth noting that corporations use long-term financing via long-term instruments in order to raise capital to finance capital investment projects. Hence, the name capital markets.

2.2. Asset Classes. Aside from the previous four conventional classifications, in practice, financial markets are usually grouped by the **asset classes** that are traded on them. Asset classes are broad categories of securities with similar characteristics. With the proliferation of financial engineering, financial instruments are becoming more complex, and some financial instruments possess characteristics of multiple asset classes, in which case they are often called **hybrid instruments**. Nevertheless, many practitioners tend to group instruments into five primary asset classes—**public equity**, **fixed income**, **money market**, **foreign exchange**, and **alternative investments**. The following is a brief description of the markets corresponding to the previous asset classes.

2.2.1. *Public Equity.* This asset class refers to the traditional stocks traded on **stock exchanges**. The largest stock exchanges are the New York Stock Exchange (NYSE), NASDAQ, and Japan Exchange Group. Other prominent exchanges include London Stock Exchange, Euronext, Hong Kong Stock Exchange and Toronto Stock Exchange. There are nearly 60 exchanges all around the world.[1] Some of the smallest stock exchanges include Mauritius Stock Exchange, Ljubljana Stock Exchange, Malta Stock Exchange and Cyprus Stock Exchange. Stocks traded on the largest exchanges are very liquid and could be bought and sold in large quantities relatively easily. Stocks listed on smaller stock exchanges trade more thinly and may be expensive to buy in large quantities. Moreover, exchanges use different languages, rules, trading hours and holiday schedules. For example, some stock exchanges prohibit short-selling of stocks.

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[1] http://www.world-exchanges.org/statistics/monthly-reports

2.2.2. *Fixed Income.* Fixed income markets trade bonds, interest rate swaps, credit default swaps and other instruments associated with debt. These markets are also sometimes referred to as **debt**, **credit**, or **interest rate** markets. In contrast to public equities, most of which trade on the exchanges, the majority of fixed income markets consist of institutions trading directly with each other—a practice known as **over-the-counter** market. Despite being over-the-counter mostly, global fixed income asset class is almost two times larger than the public equity market.[2] There are often significant minimum trade size requirements for bonds and bond trading necessitates having a good network of many counterparties who hold inventories of bonds for sale. Absence of large centralized exchanges and large trade size requirements make direct investment in bonds rather prohibitive for retail investors. This stands in contrast to the equity markets, where many individuals can participate via discount brokers with small initial investment. Usually the best way for individuals to invest in the bond market is via numerous fixed income mutual funds or exchange traded funds.

2.2.3. *Money Markets.* Money markets are OTC markets that trade in safe short-term debt instruments known also as **cash instruments** or **money market instruments**. Cash instruments can have time horizon ranging from less than a day to one year. Financial institutions and corporations lend and borrow on short-term basis from one another in the money market. In fact, money market is considered a subset of Fixed Income asset class. But it is worth mentioning separately because it is perceived to be one of the lowest risk asset classes, and therefore, it is frequently used to invest any idle principal. Money markets are generally very liquid, except during a market shock when every market participant is worried that other market participants may not be able to repay even short-term debts. The so-called **"liquidity crunch"** that happened in 2008, where money markets became significantly less liquid than usual as a result of the sub-prime mortgage crisis, is a key in point.

2.2.4. *Foreign Exchange.* Foreign exchange market, or **forex** for short, is a decentralized market where foreign currencies are traded. Since it is decentralized it is often possible to trade currencies in forex around the clock. Spurred by globalization and international trade, forex is actually the largest market in the world by daily volume with over $5 trillion traded daily.[3] The four most commonly traded currency pairs are EUR/USD, USD/JPY, USD/GBP, and USD/CHF.

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[2] http://finance.zacks.com/bond-market-size-vs-stock-market-size-5863.html

[3] http://www.reuters.com/article/2013/09/05/bis-survey-volumes-idUSL6N0GZ34R20130905

2.2.5. *Alternative Investments.* Alternative investments is a "catch-all" category. Although this asset class is the smallest compared to the other asset classes, it represents the greatest variety of markets. It includes markets like **commodities**, **private equity** and **venture capital**, **real estate**, **hedge funds**, and other markets whose performance is uncorrelated with traditional asset classes.

3. Financial Instruments

We can group financial instruments into short-term and long-term, i.e., money markets instruments and capital markets instruments. There are, of course, many other ways of classifying instruments; classification by maturity is just one of them. The purpose of this section is not to list the various classifications of securities, but to discuss their characteristics.

3.1. Money Market Instruments.

Money market instruments have maturity less than one year. Money market instruments are more liquid than capital market instruments. Since they are short-term instruments, they are considered less risky compared to capital market instruments. The longer the maturity of any financial security the more it is prone to market fluctuations as a result of financial news and other shocks. Because of their high liquidity and less risk, money market instruments are held and widely traded among bank and financial institutions. Below are different types of money market instruments.

3.1.1. *Treasury Bills (TBs).* Treasury bills are financial instruments issued by the government with maturity less than one year. Traders prefers to hold TBs as an asset because they are very safe. In fact, TBs are considered the least risky financial instruments since they are guaranteed by the government. If the government decides to pay them off, it can simply raise taxes. Thus, default is unlikely to occur. For that reason, in finance theory, the rate of return on TBs is known as the **risk-free** rate of return. It is the rate of return that any investor can get without bearing financial risk.

3.1.2. *Commercial Papers.* Commercial papers are short-term debt instruments issued by banks, corporations, and finance companies to obtain short term funds. Most commercial papers, like TBs, are issued at a discount. Only the most creditworthy banks and corporations are able to sell commercial paper. Moody's and Standard and Poor's assign credit rating to different issuers based on each issuer's risk. Consequently, commercial papers of some companies have high yield than others depending up on the rating, which is based on the risk of each company.

3.1.3. *Certificate of Deposits (CDs).* Banks also raise short-term funds by issuing CDs. At one time, CDs were non-negotiable, i.e., the original purchaser could not sell them without incurring interest penalties. Since 1961, banks have issued negotiable CDs that are traded in the secondary market. Both negotiable and non-negotiable CDs are short term instruments.

3.1.4. *Eurodollars.* Dollar deposits in banks located outside US are called Eurodollar deposits. These deposits emerged because of the highly regulated US banking industry. In order to get rid of such regulations, people transferred their money away from US to Europe to get more profit on their savings. These short-term deposits are called "euro deposits."

3.1.5. *Repurchase Agreements.* A repurchase agreement is a contract to sell a financial asset so that the seller buys back the asset at a later date, at a higher price. This means that effectively, the seller of the asset borrows from the buyer. These agreements are usually short-term.

3.1.6. *Banker's Acceptance.* Banker's acceptance is a bank loan that is used by a company to finance storage or shipment of goods. These loans are short-term since they are used to finance mainly the day-to-day operations of the firm.

3.2. Capital Market Instruments. Capital market instruments have maturity more than one year. So, financial instruments with intermediate maturity (1 to 10 years) and with long maturity (over 10 years) are included in this category. Capital market instruments are less liquid than money market instruments. Since they are long-term instruments, they are considered more risky compared to money market instruments. Below are the types of capital market instruments.

3.2.1. *Equities.* Equities are shares of ownership issued by corporations. Equities are issued in two forms: **Common stocks** (**CS**) and **Preferred Stocks** (**PS**). Owners of equities, also known as shareholders, are **residual claimants** on the income and net worth of the corporation. A residual claimant is an economic agent who has the sole remaining claim on an organization's net cash flows, i.e., on cash flows remaining after the deduction of precedent agents' claims. For instance, by examining the income statement of any corporation, you can notice that the corporation pays its debt holders first. Thus, debt holders are considered first claimants. Preferred shareholders are paid next and, finally, common shareholders are paid last.

In addition to residual profit, residual claimants also bear residual risk. This means, in case of bankruptcy, residual claimants are the last in line to claim any of the value of the liquidated assets of the firm.

Common shareholders receive **dividends**. These are periodic payments made to holders subject to the firm's dividend policy. Because corporations are ongoing concern as they remain profitable, the equities that they issue have no stated maturity. Therefore, common stocks don't have maturity date and don't offer fixed payments; rather they offer dividends. Common stocks are shares issued by corporations that give the right to its owner to vote on management issues but offer no guarantees of dividends or of market value of the firm in case of bankruptcy. Holders of preferred stocks, on the other hand, have no voting rights. They sacrifice this right in exchange for a guarantee that they will receive dividends and if the company goes bankrupt, they are first claimant over CS holders. Thus, preferred shares are shares issued by corporations that give the right to its owner to receive a guaranteed **fixed** dividend payment **for ever**, but offer no voting rights. Notice that a preferred stock is considered a hybrid security; it has a fixed payment, which is bond feature, and it has no maturity, which is a CS feature.

3.2.2. *Corporate Bonds.* Corporate bonds are long term debt instruments issued by corporations to raise funds. A bond is an agreement between the issuer, i.e., the corporation, and the holder, i.e., the saver/lender/investor, by which the issuer is entitled to pay a fixed amount, known as the coupon payment, every specific period of time till maturity. Bonds are less risky compared to CS as the holders are first claimants. However, returns on bonds are lower compared to stock returns. Bonds have a stated maturity date whereas equity instruments don't.

3.2.3. *Treasury Bonds.* Are long-term bonds issued by the government to raise funds. It is a long-term form of government borrowing to finance government budget deficit.

3.2.4. *Mortgages.* Mortgages are long term loans made to businesses or individuals for purchases of land and buildings, i.e., real estate.

4. Financial Intermediaries & Institutions

A **financial intermediary** is an organization that raises money from savers (also known as lenders) and provides financing for borrowers (individual, companies, and other organizations). Financial intermediaries raise money by selling units to investors. These funds are then pooled and invested in a portfolio of assets. Common financial intermediaries are mutual funds, exchange-traded funds, hedge funds, private equity funds, and pension funds. On the other hands, a **financial institution** is an intermediary that does more than just pooling and investing. They provide additional financial services. Common examples of financial institutions are banks and insurance companies.

Before we discuss the different types of intermediation, it is worth noting the main reasons behind the existence of financial intermediation: Why do many savers choose to hold their savings at a financial intermediary? The key argument is related to the existence of **asymmetric information** in financial markets. Whenever one party of any transaction has information not possessed by the other party, asymmetric information exists. Asymmetric information is common in financial markets. For that reason, investors tend to revert to financial intermediaries to deposit and invest their savings. A financial market characterized by asymmetric information is said to be inefficient in the sense that profit opportunities could be exploited by those who have information advantage. For that reason, financial markets are regulated by government agencies and regulatory authorities to ensure the transparency of financial transactions. For instance, the U.S. Securities and Exchange Commission (SEC) is the government agency that ensures the efficiency of financial markets in the U.S. The Financial Consumer Agency of Canada (FCAC) is responsible for informing and protecting consumers of financial products and services. We will explore the notion of efficient markets in detail in Chapter 5.

In addition to asymmetric information, there are other benefits to financial intermediaries. Financial intermediaries collect information on behalf of savers. They can gather information about the financial instruments and monitor the performance of those who issue such instruments. This abundance of information allows financial intermediaries to provide financial services at a cost lower than what investors would have incurred if they managed their own portfolios.

4.1. Financial Intermediaries. The most common financial intermediaries are **mutual funds** and **exchange-traded funds** (**ETFs**). These intermediaries raise funds by selling individual investors units of investment. The funds are pooled and invested in a portfolio of securities. In mutual funds, investors contribute to the portfolio by buying units in it and gain a percentage of the portfolio's subsequent dividends and price appreciation. At any time, investors can cash out their investments by selling their units back to the fund. The AIM Canadian Premier Fund, for example, had about $827 million in total assets in January 2008. It held a portfolio of 77 stocks, 56 percent Canadian and 34 percent international.[4] By contrast, in an ETF, investment units are purchased on a stock exchange. For instance, the iShares CDN Bond Index Fund ETF (XBB) trades on the Toronto Stock Exchange.

Investors, usually retail traders, are attracted to mutual funds or ETFs because they offer professional management of a portfolio of assets at a low-cost. Thus, it is more efficient for most investors to buy a mutual fund or an ETF than to assemble a diversified portfolio of stocks and bonds.

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[4] Source: Brealey et al (2009).

To attract investors, fund managers strive to construct portfolios of assets that can generate more than average rate of return. In exchange for their services, fund managers take a management fee. Mutual fund also incur some expenses to run the fund. The fees and expenses vary from one fund to another. In Canada, there are over 2000 mutual funds and 60 ETFs. These funds pursue a variety of investment strategies. The process of asset allocation and portfolio construction and the various investment strategies pursued by fund and portfolio managers are at the core of finance theory. We will explore this topic in detail in Chapter 4.

A **private equity fund** is another type of financial intermediary. This type of funds invest in the equity of privately owned businesses. These businesses are usually either growing or troubled companies that don't have publicly traded equity in the stock market. Private equity provide financing in order to help these businesses to achieve stable growth. Common forms of private equity investments include venture capital and angel investing.

A **pension fund** is an investment plan set up by an employer on behalf of the employees for their retirement. It is another form of pooling and investing employee's savings. Usually, a percentage of each employee's monthly paycheck is contributed to the pension fund. The money is pooled and invested in securities of mutual funds. Pension plans are usually long-term plans that accumulate returns over the employment period so that when individuals reach the retirement age they can use the balance to finance living expenses.

4.2. Financial Institutions. A financial institution, such as a commercial bank or an insurance company, is an intermediary that does more than just pool and invest savings. Commercial banks accept deposits from savers and make loans to individuals, businesses, and other corporations. By doing so, banks don't only invest in securities, but they also loan money directly to individuals and other businesses.

Similarly, insurance companies provide services more than just pool and invest savings. They issue policies (life, property, casualty), which promise to compensate the holder for damages suffered as a result of bad events in exchange of premiums collected from policy holders. The insurance companies then invest the money received and gain profit.

5. The Corporate Environment

In the previous sections we took a macro perspective to finance and discussed in detail the financial system in general. In this section, we switch to a more micro discussion of finance, In particular, we discuss the role of financial manager in making investment and financing decisions. We emphasize the difference between both decisions and highlight their relations to the firm's financial statements. We also discuss the different legal structures of a business; namely, sole proprietorship, partnership, and corporation. We analyze the advantages and disadvantages of each type.

5.1. Basic Forms of Financial Organizations.

There are three legal forms of business organizations; namely, sole proprietorship, partnership, and a corporation. In some situations, there exist be a hybrid form of partnership and corporation. Each structure has implications regarding the liabilities of its owners, the amount of income tax paid, and the regulations governing the business. Each structure has its strengths and weaknesses. In this section, we will analyze each structure in detail.

5.1.1. *Sole Proprietorship.*

Definition 5. *A sole proprietorship is a business owned by one person.*

As the sole proprietor of the business, the owner of the business bears all the cost and reaps all the profits (as well as losses).

The key advantages of a sole proprietorship are as follows: (1) It is easily and inexpensively formed. (2) It is not subject to rules and regulations governing it. (3) It is not subject to corporate income taxes; the owner pays only income tax on the profits made by the sole proprietorship. On the other hand, this business structure suffers from some weaknesses. The main disadvantage is that the owner has **unlimited liability**; that is, the owner of the business must satisfy the business debts and other liabilities. So, in bad conditions when the business is not able to repay its debts, the debt claimants have a claim against the owner's personal assets. Another disadvantage is that it is difficult for the sole owner to raise large sums of capital. For that reason, this structure is well suited for a small business with informal structure.

It is worth noting that, in this business structure, the life of the business is limited to the life of the person who created it.

5.1.2. *Partnership.*

Definition 6. *A partnership is a business owned by two or more people and operated for profit.*

Most partnerships are established by a legal written contract that identifies the obligations of each partner. The partnership agreement also identifies how management decisions are made and the share of each partner in the firm's profits. In a general/regular partnership, all partners have unlimited liability. In a limited partnership, one or more partners can have limited liability as long as at least one partner has unlimited liability.

Like sole proprietorship, partnerships have the disadvantage of unlimited liability. If the business runs into financial difficulties, e.g., the partnership goes bankrupt, the partners might lose all their assets, even those assets that are not invested in the business. It is worth noting that each partner has unlimited liability for **all** business debts and not just his/her own share. Thus, it is crucial to know and wisely choose your partner if you are planning to adopt that type of business structure. Another disadvantage of partnerships is the difficulty to achieve large scale operations. It could also be difficult to liquidate a partnership.

On the other hand, the advantages of this business structure are as follows: (1) It can raise more funds than sole proprietorship. (2) Having more than one owner enhances the borrowing ability of the firm. (3) It is not subject to corporate income tax.

Like sole proprietorship, the life of a partnership is limited to the life of its partners.

5.1.3. *Corporation.*

Definition 7. *A corporation is a legal entity created by law and owned by shareholders who have limited liability.*

A corporation is a legal entity created by law. It has the power of an individual in the sense that it can sue and be sued. The owners of a corporation are called **shareholders** or **stockholders**. They have limited liability, i.e., the owners are not personally responsible for the corporation's obligations.

The key advantage of a corporation, as noted above, is that the owners/shareholders have limited liability. This guarantees that they can't lose more than their share in the corporation As a business structure, the corporation enjoys other advantages. For instance, it is a permanent entity, and thus, it is not dissolved by the death of its owners. It is easy to transfer ownership in a corporation, and it is easy to raise funds to finance assets.

On the other hand, a key disadvantage of this business structure is that it is subject to double taxation. The earnings of the corporations are subject to corporate tax, and the earnings paid out to shareholders as dividends are subject to income tax. The process of establishing a corporation is more expensive compared to other types of business. Moreover, corporations are subject to greater government rules and regulations as opposed to partnerships or sole proprietorships.

Another disadvantage of a corporation is the so-called "**agency problem.**" This problem arises from the fact that managers in a corporation place their personal goals ahead of the main corporate goal, which is to maximize the shareholders' wealth. Managers might lack personal interest in the corporation. They might be concerned with other goals, e.g., job security, life style, and fringe benefits. The agency problem arises when managers work for their personal goals and not for the sake of shareholders. Of course, there are means to resolve or at least minimize this problem. The threat of firing by large shareholders who are able to exert pressure on management to perform or be fired is an example. Another solution could be encourage managers to buy shares in the corporation and become themselves shareholders. Other incentives can be used to encourage managers to act in the best interest of the owners such as compensation packages and bonuses.

A corporation could be either a public company or a private company. A **public company** is a corporation whose shares are listed on a stock exchange for trading. A **private company** is a corporation whose shares are privately owned by a small group of investors and are not traded on any stock markets.

A corporation, as a legal entity, can in principle live for ever. One of the oldest corporations in Canada is Hudson's Bay company, which was established in 1670. The company was formed to profit from the fur trade between northern Canada and England by sea via Hudson's bay. It was a public company until, recently in 2006, when Jerry Zucker purchased all of its shares in a takeover. Hudson's bay is considered now a private company.

5.2. The Role of the Financial Manager. The main duty of the financial manager of any firm is to manage the firm's capital. To this ends, the financial manager ought to be able to identify how much capital is needed, from where this capital can be raised, and how to spend it. All the previous tasks of identifying the sources and uses of funds and how to raise funds should be done in order to achieve one ultimate objective, which is **maximizing the current value** of the company's shares and the wealth of its shareholders. Shareholders are always after maximizing their wealth. Therefore, they are looking for managers who can make decisions that maximize the current value of the firm. This, in turn, will maximize the wealth of the shareholders. But, what about profit maximization? It sounds reasonable for a manager to seek its company to be more profitable. However, profit maximization is not a well-defined corporate objective. Brealey et all (2009) gave the following three reasons:

(1) Maximizing profit as an objective doesn't specify which year's profits. A company may be able to increase current profits by cutting back on maintenance and staff training for instance. This plan, however, might not be welcomed by shareholders because it could potentially damage profits in future years.

(2) Maximizing profit may entail cutting current dividends and investing the freed-up cash in the firm. Although this strategy may lead to more profit in the future, it might not be in the shareholders' best interest if the company earns only a low rate of return on the extra investment.

(3) Different accountant may calculate profits in different ways. So, a decision to increase profits using one set of accounting rules may reduce them using another.

In short, the main goal of a corporation is to maximize current market value, and by performing a variety of tasks, financial managers are expected to help shareholders realize this objective. Our next task is to discuss the duties of financial managers in this regard.

In their quest to maximize the current value of a corporation, financial managers are keen to make good **investment decisions** and **financing decisions**.

Investment decisions, also known as **capital budgeting decisions**, are decisions regarding which real assets the firm should acquire. The decision to invest in a project or a number of projects starts with identifying the prevailing investment opportunities. The financial manager helps the firm to identify which projects are promising and how much to invest in each project. Investment decisions are often called capital budgeting decisions because they are reflected in the firm's annual budget statement. By making investment decisions, the financial manager is engaged in determining the mix of assets to be invested, i.e., short term investments in current assets or long-term investments in fixed assets. We will discuss capital budgeting and capital budgeting techniques in more detail in Chapter 3.

In addition to investment decisions, financial managers need to make financing decisions. A **financing decision** is the decision of how to raise funds to pay for investments in real assets. In general, there are two ways to raise funds: either through debt or through equity. Debt financing refers to raising funds via debt instruments, i.e., by taking a loan or by issuing bonds. Equity financing refers to raising funds by issuing stocks. It is worth noting that financing decisions are related to the firm's liabilities and equities in its balance sheet. The balance sheet is a summary of the firm's financial position at any given point in time. The statement balances the firm's assets (what it owns) against its financing, which can be either debt (what it owes) or equity (what was provided by the owners).

Financing decisions could be short-term or long-term. Short-term loans and short-term bonds are examples of short-term debt financing. Long-term bonds and issuing common stocks are examples of long-term financing.

In addition to investing and financing decisions, the financial manager is also involved in performing financial analysis and planning. This task involves transforming financial data into a form/measure that can be used to assess the firm's financial performance. This function makes use of **financial ratios** in assessing the firm's liquidity, profitability, operational efficiency, and solvency.

6. Practice Problems

(1) Give an example of a money market instrument and an example of a capital market instrument.
(2) What is the difference between primary and secondary markets?
(3) What is the difference between debt and equity markets?
(4) What is the difference between Exchange and OTC markets?
(5) What is the main objective of shareholders? What is the role of the financial manager?
(6) What is the agency problem?
(7) Explain how funds are channelled in the financial system.
(8) The shareholder dilemma: The focus has always been on maximizing shareholders' value at the cost of the stakeholders. This leads to negative effects on the economy and the job market. Suggest one solution to this problem and explain your reasoning.

CHAPTER 2

Risk and Return

From a broad perspective, financial theory (and finance in general) is based on **two** key ingredients that are essential to the very nature of finance: **time** and **risk**. Finance is all about time. Any investment decision such as buying a stock, or any asset in general, involves paying an amount on the spot to acquire the asset in hope of achieving financial gain when you sell the asset at a future time. Thus, the whole financial plan is based on **time**; when the time element is not present, the future doesn't exist, and consequently, investors have no motive to buy assets in the first place. The value of time is captured by discounting all expected future cash flows from holding an asset at an appropriate discount rate. This exercise is known as **financial valuation**. We will explore this notion in detail in Chapters 4, 5, and 6.

Once we admit that time is indeed a main factor in making any financial decision, we implicitly recognize the second main factor in making investment/financial decisions; **risk**. Since the future is by essence **uncertain**, then making any financial decision that entails a payout in the future is necessarily risky. Both elements; time and risk, are non-separable, and together they constitute the basis of **finance theory** in general and asset valuation in particular.

This chapter serves as an introduction to finance theory. In particular, in this chapter, we will learn how risk and return are **defined** and **measured** in finance. The risk-reward or risk-return notion is the building block of finance theory. We will discuss finance theory in Chapter 3. In particular, we will focus on one particular financial model of price formation; the so-called capital asset pricing model. But, in order to fully understand this model, one should begin with understanding the risk and return on financial assets. This is the subject matter of this chapter.

1. Preliminaries: Returns as Random Variables

In this section, we review some fundamental concepts that will be proven useful in our quest to define and measure risk and return pertaining to financial assets. Since future asset returns are unpredictable, i.e., random. We ought to first define formally what it means by a random variable. Once this definition is clear, the future rate of return on any asset will be treated as a random variable whose values are probabilistic, i.e., determined by chance. The advantage of seeing the rate of return on a financial asset as a random variable is that this setup allows us to compute the average of the probabilistic returns pertaining to the asset and the fluctuations of these returns around that average. The average rate of return is a measure of the **expected rate of return** on the asset and the sum of the fluctuations of returns around this average is a measure of the volatility or the **risk** of the asset.

We begin by giving a brief discussion on the process of assigning numbers to meanings or the so-called **measurement** in statistics. But, before we proceed, perhaps we should begin by defining what it means by the word "statistics."

Statistics is a branch of applied mathematics that uses mathematical techniques for summarizing quantities of information and displaying them in an informative way. The word "statistic," in singular form, has two meanings. It might refer to information to which the mathematical procedures of statistics are applied. For example, the number of recorded accidents on a high way is a statistic. The second definition of statistic, which is used in science, is the number resulting from the application of the mathematical procedure to a body of information. An example is the average monthly number of accidents on a high way. The latter definition is the one that we are interested in.

Gathering data and information has many purposes. The basic purpose of gathering information is **description**. The purpose here is to describe what is happening in a summary form. This is referred to as **descriptive statistics**. Another purpose of gathering information is **evaluation**. In academia, for instance, educators use statistical methods to evaluate the teaching performance of their instructors. A third purpose of gathering information is **prediction**. A typical example is forecasting the students' performance based on their testing scores. Here collecting data on testing scores serves to predict the performance of future students.

In order to apply mathematical procedures to gathered information, it has to be in the form of numbers. Some gathered information is in the form of verbal descriptions or answers to a set of questionnaires and, therefore, need to be converted into numbers in order to apply the statistical procedures necessary for the analysis. The process of assigning numbers to meanings is known as **measurement** and those resulting numbers that have meanings are called **data**.

Measurement, or assigning numbers to objects or events, is done according to a set of rules. There are many measurements that we frequently encounter such as weight, height, distance, time, and so on. The set of rules are referred to as the **scale** that makes the numbers assigned to any measurement meaningful. Consider, for example, the following statement: the distance between point A and point B is 20 kilometers. The measurement here is the distance and the set of rules, or the scale, here is the kilometers. Without assigning this reference to the number 20, it will be meaningless. Assigning the scale is crucial in obtaining accurate information. Sometimes, we have to define the scale ourselves through certain rules. Measuring the gender of workers in a factory is a key in point. One way to do that is to assign the number zero to males and the number 1 to females. The scale here is that males receive zero and females receive 1.

Collecting data could be done either via designing a survey or by running an experiment a number of times and recording the observations on a particular event. When studying the characteristics of an event, it is important to notice whether those characteristics take more than one scale value or same value every time they occur. They are called **variables** if their values vary and **constants** if they take the same scale value every time they occur randomly in an experiment. Therefore, we can define a **random variable** as a number whose scale value is determined by chance. For notational purposes, we will use letters to denote random variables and letters with subscripts i or t to denote their realizations/observations. For instance, if the letter x refers to a specific random variable, then x_i or x_t will refer to a particular realization/observation of the random variable x in a particular experiment. If the experiment is repeated n times and the random variable x takes a specific value each time the experiment is repeated, then x_i, for $i = 1, 2, ..., n$ are the realizations of x in this experiment.

A data set is a set that contains a number of realizations/observations. For instance, the data set

$$A = \{x_1, x_2, x_3\}$$

consists of three realizations on the random variable x. The size of this set $n = 3$.

Definition 8. *A random variable is a number whose scale value is probabilistic, i.e., determined by chance.*

At this point, it is crucial to make the distinction between **population** data sets and **sample** data sets. A population data set is a complete collection of observations we want to study. It is usually difficult to define since a population data set contains all members of the study. A sample data set is a subset of a population set that contains part of the population data. An **observation unit** (element) is an object on which a measurement is taken.

Example 1. *For instance, consider conducting a survey to find the average weight of individuals living in the city of Victoria. In this study, the observation unit is an individual and the random variable is the weight. The* *population data set includes the weight of each and every individual living in Victoria. If we let* x *denote the weight and* N *be the number of individuals living in Victoria, then the population data set is a set* $P = \{x_1, ..., x_N\}$ *that consists of* N *observations. If a sample of* n *individuals,* $n < N$*, is taken from the population, then the sample data set, denoted by* S*, is a subset of* P *and consists of the weights of the* n *chosen candidates. For instance, if individuals 6,9, and 11 are chosen and their corresponding weights are recorded in* S*, then*

$$S = \{x_6, x_9, x_{11}\}$$

is the sample data with $n = 3 < N$*.*

Here is another example.

Example 2. *Suppose you want to measure the average amount spent on advertising by firms in your industry. The observation unit in this case is a typical firm. The random variable is the dollar amount spent on advertising. The population data set consists of all amounts spent on advertising by each and every firm in the industry. The sample data set consists only of the number* n *of selected firms from the total population.*

It is worth noting that obtaining information on every observational unit in a population is almost impossible. In fact, if we have data on every member of a population, then we don't need to take a sample to study it. We can directly use the population data to conduct our analysis. However, because it is difficult and expensive to collect data on every member in the population, we ought to draw a good sample and use it to deduce a general conclusion about the population. A good sample means a well representative sample that is free from **bias** and **measurement errors**. For instance, if we want to compute the average household consumption in an economy, we ought to draw a representative sample of households that consists of families from different income classes, i.e., low, middle, and high income. If the majority of families in the chosen sample fall in the high income class, the sample is said to be biased for it doesn't represent the population. A biased sample lead to erroneous results and forecasts. After all, we are studying the sample to infer something about the behavior of the entire population.

Now, since we made the clear distinction between a sample and a population, our next task is to discuss what exactly we can do with the sample data. As mentioned above, sample data is used to infer some characteristics of the population. There are many indices that are useful to describe the characteristics of any set of data. Two indices are particularly useful in finance: (1) Indices of central tendency (or location) and (2) Indices of dispersion (or variability). The former are measures of how the data tend to cluster, i.e., center, about certain numerical values and the latter are measures of how the data is spread about its central tendency. We will focus our attention on the **arithmetic mean** as a measure of central tendency and the **standard deviation** as a measure of variability.

2. Sample Mean, Population Mean, and the Rule of Expectations

The arithmetic mean (or simply the mean) of a random variable x is the most common measure of central tendency or location. It is denoted by $\overline{x}$ if the data are collected from a sample. If, on the other hand, it happens that we have data on the entire population, then the mean of the random variable x will be denoted by the Greek letter μ. Sometimes, we use the notation μ_x, when we have more than one random variable and we want to emphasize that the mean belongs to x.

As we mentioned in the previous section, it is difficult to have a population data set. For that reason, researchers, in practical applications, work with sample data. Therefore, in solving problems or analyzing sample data in practice, we should use the notation $\overline{x}$ to express the mean or the random variable x. On the other hand, the notation μ is reserved for theoretical purposes, where we adopt the implicit assumption that the true population is known to us. Now, since we are familiar with the notation, we proceed to define the sample mean.

2.1. The Sample Mean.

Definition 9. *The **sample mean** of a random variable x, denoted by $\overline{x}$, is the arithmetic average of the total number of observations, n, in the sample data set $S = \{x_1, x_2, ..., x_n\}$. It is defined by the following equation*

$$\overline{x} \overset{defn}{=} \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{2.1}$$

Example 3. *Suppose you recorded the end-of-day closing price of a particular stock in the last week and obtained the following data set of prices*

$$\{4, 5, 1, 8, 2\}.$$

It is obvious that the previous set is a sample set of size $n = 5$. So, if we let x denote the price of that stock, then the sample mean/average of the above 5 observations is

$$\overline{x} \overset{defn}{=} \frac{\sum_{i=1}^{5} x_i}{5} = \frac{4 + 5 + 1 + 8 + 2}{5} = 4.$$

In the previous example, we take the sample average of prices. In practice, however, we care more about rates of returns rather than just prices. The rate of return on any asset at time t, which we will denote by r_t in the text, is computed by the conventional percentage change formula; that is,

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100\%, \tag{2.2}$$

where P_t is the current period price and P_{t-1} is the lag price. For instance, if the end-of-day closing price on Apple stock is $P_t = \$220$ and yesterday's closing price is $P_{t-1} = \$200$, then by holding this stock for one day you achieved a rate of return of 10%. This is easy to verify from equation (2.2) as

$$\begin{aligned} r_t &= \frac{P_t - P_{t-1}}{P_{t-1}} \times 100\% \\ &= \frac{220 - 200}{200} \times 100\% \\ &= 10\%. \end{aligned}$$

It is worth noting that the same result could be applied for any frequency, i.e., for weekly prices, monthly prices, or annual prices.

EXAMPLE 4. *Suppose you gathered a sample of monthly prices of XYZ stock over the period between June 2010 and November 2010. The data is shown in the table below, where P_t denote the stock price at time/month t. Compute the average rate of return, i.e., the expected rate of return, on XYZ.*

$t =$ *month*	P_t (\$)
June	10
July	12
August	15
September	12
October	15
November	18

First, note that this is clearly a ***sample*** *of closing prices that consists of $n = 6$ observations. Then, if we let r be rate of return as a random variable, r_t is a realization at time t computed as*

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100,$$

then the ***sample mean*** *rate of return on the stock, which is the average rate of return, can be computed from the conventional sample mean formula as*

$$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n}.$$

Table 1 shows how r_t is derived from the price data. Notice that the first observation is lost since the price data on the month of May is not available. Thus, the workable sample number of observations for the rate of return r_t is 5 instead of 6 observations. Therefore, the sample mean rate of return on XYZ stock is

$$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{70}{5} = 14\%.$$

So, based on the historical data provided, the average rate of return on the stock is 14%.

t = Month	P_t (\$)	P_{t-1} (\$)	$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100$ (%)
June	10	NA†	NA†
July	12	10	20
August	15	12	25
September	12	15	−20
October	15	12	25
November	18	15	20
			$\sum_{t=1}^{n} r_t = 70$
$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{70}{5} = 14\%.$			
†Not Available			

TABLE 1. The sample mean rate of return on XYZ stock over a period of 5 months.

EXAMPLE 5. *The following is a* ***sample*** *of several days' rates of return on AAA stock. What is the average rate of return on this stock?*

Month	*return (%)*
1	+1
2	−1
3	−2.5
4	−0.5
5	+2
6	+1
7	+4
8	+2

Notice here that we have historical data on AAA returns. In particular, we have a sample of size $n = 8$ observations. Also note that we already have returns, so no need to convert prices into returns as we did in the previous example. We can proceed immediately and compute the sample average as follows:

$$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{+1 - 1 - 2.5 - 0.5 + 2 + 1 + 4 + 2}{8} = \frac{6}{8} = 0.75\%.$$

Thus, the average rate of return on AAA is 0.75%.

We saw in the previous example how can we find an average return from past historical data. We found that $\overline{r} = 0.75\%$ in the previous example. But, what does this number mean in finance? Well, this is one way investors or financial analysts formulate their expectations regarding asset returns. In this particular method, investors make use of previous historical data to compute an average measure of returns. This measure could be taken to be the **expected** or **required rate of return** on a particular asset. It is the rate of return that the investor would required in order to invest his or her money in that particular asset. We will elaborate more on this point in Section 3. For now, we give the following definition.

DEFINITION 10. *Given historical sample data on asset prices (or rates of return), the expected/required rate of return on an asset is the sample average of returns and it is calculated from the conventional sample mean formula in equation (2.1).*

PROBLEM 1. *Suppose you gathered the following end-of-day closing prices on XYZ stock. Based on this sample, what is the rate of return that you would require as an investor in order to invest in XYZ stock?*

$t =$ *day*	P_t (\$)
1	8
2	10
3	14
4	16
5	14

SOLUTION 1. *Notice that this is a sample problem and we need to compute the expected or required rate of return on XYZ stock. First, we need to convert prices to rates of return as follows:*

$t =$ *day*	P_t (\$)	P_{t-1} (\$)	$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100$ (%)
1	8	$NA^\dagger$	$NA^\dagger$
2	10	8	$\frac{10-8}{8} \times 100 = 25\%$
3	14	10	$\frac{14-10}{10} \times 100 = 40\%$
4	16	14	$\frac{16-14}{14} \times 100 = 14.3\%$
5	14	16	$\frac{14-16}{16} \times 100 = -12.5\%$

The expected or required rate of return on XYZ stock is the sample average, which is calculated as follows:

$$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{25 + 40 + 14.3 - 12.5}{4} = \frac{91.8}{4} = 22.95\%.$$

Thus, the investor would require approximately 23% rate of return in order to be motivated to invest in it.

In the previous examples, we saw how investors' expectations regarding rates of return are formulated based on sample historical data. The expected rate of return is simply the sample mean. What about μ; the population mean? In the following section, we discuss the meaning of the population mean and how it differs from the sample mean. We also explain how the population mean can be used in finance to compute the expected or required rate of return on any asset.

2.2. The Population Mean and the Expectation Operator. If we entertain the assumption that we have all possible values taken by the random variable x in every state of nature, how can we express its mean? To answer this question, we ought to discuss the rule of **expectation** and the so-called "E" operator.

The expectation of a random variable x is denoted by Ex, where the expectation E is an operator that acts on the random variable x. The concept of expectation is most easily understood in economics and finance. For instance, suppose the random variable under consideration is the rate of return, r, on a particular stock at different states of the economy. For simplicity, assume two states: a recessionary state and an expansionary one. Suppose that the stock is predicted to achieve 2% if the economy moves into a recession phase and 8% if the economy expands. Assume further that the two states of the economy are equally likely to occur, i.e., the probability of a recession equals to the probability of an expansion equals to 0.5. Now, the question is: What is the **expected** rate of return that this stock is **likely** to achieve? In other words, what is the **most likely rate of return** that will be realized on this stock? Since there is a 50% chance that the stock will achieve 2% rate of return if the economy moves into recession and 50% chance that it will achieve 8% if the economy expands, then the most likely outcome is a kind of a **weighted average** of the two expected rates of return, i.e., the most likely outcome is $(2\% \times 0.5) + (8\% \times 0.5) = 5\%$. This is the **expected value** or the **mean** of the random variable r, which, in statistics, is considered to be a measure of the "**center**" of the corresponding probability distribution, i.e., a measure of central tendency. It is worth noting that, in this example, although we assumed that there are only two possible values taken by the random variable r and two corresponding probabilities, we are still dealing with a population data. We simply assumed that **all** the random states of nature are 2 in total. Also, note that it is always true that the sum of all probabilities adds to 1. In short, if we happen to know each and every probabilistic realization of a random variable in all states of nature, i.e., we have population data, then we can compute the **mean** of this random variable by following the **rule of expectation**; the sum of each realization times its corresponding probability. Here is a formal definition.

Definition 11. *Consider n states of the economy and let $p_1, p_2, ..., p_n$ be the state probabilities such that $p_1 + \cdots + p_n = 1$. Let $x_1, x_2, ..., x_n$ be all realizations on a random variable x in all states. The* ***population mean*** *of a random variable x, denoted by μ, is its expected value, Ex, and is defined by the equation*

$$Ex \overset{or}{=} \mu \overset{defn}{=} x_1 \times p_1 + \cdots + x_n \times p_n. \tag{2.3}$$

Definition 12. *The rule of expectations states that the expectation of any random variable x is just the sum of each realization or observation on x times its corresponding probability of occurrence.*

At this point, the reader might wonder about the connection, if any, between the sample mean, $\overline{x}$, and the population mean Ex? The sample mean is computed from a small sample of size n whereas the population mean is the expected value of the random variable, which is computed under the presumption that we have knowledge of the population data set,−a **large** set of size N.[1] The two notions are measures of central tendencies that are computed using different data. The sample mean uses historical sample data whereas the population mean uses all probable data on the random variable under consideration. It is worth noting that, although the two notions are calculated differently, they are in fact equivalent. It can be shown that when the sample is large enough the sample mean approaches the population mean. Proving this last statement is beyond the scope of this book. However, for our purpose, it suffices to know how to compute both measures and how to explain their meanings.

Example 6.

State of the Economy	*Return*	*State Probability*
Recession	$r_1 = -0.2$	$p_1 = 0.2$
Steady State	$r_2 = 0.2$	$p_2 = 0.5$
Expansion	$r_3 = 0.5$	$p_3 = 0.3$

Expected returns at every state and the corresponding probabilities

Suppose your analyst provided you with the above data on the returns pertaining to AAA stock. Assume that the economy is characterized by three states: recession, steady state, and expansion. Your analyst gathered some data regarding the expected annual returns on this stock in each state of the economy with the corresponding probability of occurrence. What is the expected rate of return, Er, on this stock? Notice that, in this example, the data reflects the various states of the economy and, thus, it clearly represents the ***population***. *Therefore, the mean or expected rate of return on the stock is simply the population mean μ and is computed as follows:*

$$\begin{aligned} Er \overset{or}{=} \mu &= r_1 \times p_1 + r_2 \times p_2 + r_3 \times p_3 \\ &= (-0.2)(0.2) + (0.2)(0.5) + (0.5)(0.3) = 0.21 \text{ (or } 21\%\text{)} \end{aligned}$$

So, what do we mean by $Er \overset{or}{=} \mu = 21\%$ in the previous example? It means that the average rate of return, based on all probabilistic states of the economy, is 21%. In the same way as we did with the sample mean, we can also take this measure to represent the investor's **expected** or **required**

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[1] In practical applications, a small sample consists usally of 30 or less observations. A sample is considered large enough if it has more than 30 observations.

rate of return on the asset. Thus, in the previous example, the investor's expected or required rate of return to invest in AAA stock is 21%.

Definition 13. *Given probabilistic data on asset prices (or returns) in every state of the economy, the expected/required rate of return on an asset, Er, is the population average of returns and it is calculated from the conventional population mean formula in equation (2.3).*

3. Return: The Meaning of the Expected Rate of Return

If the rate of return, r, on a stock or any financial asset is seen as a random variable that has random probabilistic realizations in different states of the economy. Then, the population mean (or population average) is the **expected rate of return**, Er, on that particular asset. Of course, if we have historical data on the rates of return pertaining to the asset, we can still compute the sample average, $\overline{r}$, and this average can be taken to be the expected rate of return on the asset.

The previous two methods are simply the two ways to compute the expected rate of return on any asset in practice. But, what about the meaning of the resulting values? What does it mean that the expected rate of return on a stock is, say, $Er = 5\%$? From the investor point view, this is the **most likely** rate of return that will be realized on this stock if s/he decided to invest in it. Obviously, this expected rate of return is calculated based on the investor's beliefs/expectations about the future states of the economy and the financial markets. Investors are interested in these calculations to either assess how much return on average they will achieve on their investment or to assess the rate of return they would require on a similar asset in the same risk class.

It is worth noting that, in finance, the expected rate of return on a stock is also called the **required rate of return** (RRR) for it is the rate of return that the investor would required to invest in that particular asset. For instance, suppose you want to invest in Apple stock. What is the RRR for you as an investor to be motivated to put your money in this particular stock? There are many ways to come up with the RRR on Apple stock. (1) If you have historical data on Apple stock price, you can compute the sample mean, $\overline{r}$, and take that number to be the expected or required rate of return on Apple stock. For instance, if $\overline{r} = 12\%$, then you expect 12% on that stock or you require 12% on that stock to invest your money in it. (2) If you have future probabilistic returns on Apple stock, then you can compute the population mean, Er, and take that number to represent the expected or required rate of return on that particular stock. (3) If you don't have any data on Apple, but you have data on a similar asset in the same risk class, e.g., Miscrosoft stock, then you can compute the sample mean or the population mean and take that number to be the required rate of return on Apple stock. There is also another way to compute the RRR on any stock; namely, by using an equilibrium model of price formation from finance theory. There are many equilibrium models. We will focus on one particular equilibrium model of price formation known as the capital asset pricing model and we will discuss it in detail in the following chapter. But, for now, the reader should be clear about the equivalent meaning of the terms **required rate of return** and **expected rate of return**, and how they can be computed in practice.

Definition 14. *The required rate of return (RRR) on an asset is its expected rate of return, Er, which is computed as the population mean, μ, or the sample mean, $\overline{r}$, of the rates of return on that particular asset. It could also be taken as the required or expected rate of return on another asset in the same risk class.*

Problem 2. *Suppose you are interested in investing your money in TD stock, which is currently traded at the Toronto Stock Exchange (TSX). You want to compute the expected rate of return on that stock. But, for some reason, your analyst wasn't able to find any rigorous predictions on TD's behavior in various states of the economy. Your analyst, however, provided the following data on RBC; another asset in the same risk class:*

State of the Economy	*Return*	*State Probability*
Recession	$r_1 = -0.1$	$p_1 = 0.2$
Steady State	$r_2 = 0.1$	$p_2 = 0.5$
Expansion	$r_3 = 0.2$	$p_3 = 0.3$

Expected returns at every state and the corresponding probabilities

How can you use the previous data to calculate the expected or required rate of return on TD stock?

Solution 2. *Since we don't have population data on TD stock, we can still use RBC data (or even the industry average data) to find a good proxy of the expected rate of return on TD. As long as the asset under consideration is within the same risk class as RBC, the calculated RRR or Er should be close enough. Remember, we are not looking for accuracy here; rather, we are looking for a good indicator of the most likely return that the investor would achieve on TD. Thus, in this case,*

$$\begin{aligned} RRR_{TD} &= Er_{TD} = Er_{RBC} = r_1 \times p_1 + r_2 \times p_2 + r_3 \times p_3 \\ &= (-0.1)(0.2) + (0.1)(0.5) + (0.2)(0.3) \\ &= 0.09 \text{ (or 9\%).} \end{aligned}$$

4. The Variance of a Random Variable

As we have seen in the previous section, the expected value (or the mean), whether it is computed from the sample or from the population, provides us with information concerning the central tendency (or the location) of a distribution. But, it does not convey any information regarding the **dispersion** of the distribution. The dispersion of a distribution is a measure of all deviations from its central tendency. In finance, it is considered a measure of **risk**. To understand the concept of dispersion or deviations from the mean of a distribution, perhaps it is best to give an example that illustrates the rationale behind it, then proceed with a formal definition.

EXAMPLE 7. *Suppose you recently invested a portion of your wealth in the stock of ABC corporation. As a typical investor, you care about the rate of return on your investment. Let* r *denote the rate of return on ABC stock. A realization on* r *at time* t*, denoted by* r_t*, is computed by knowing the stock price at time* t*,* P_t*, and its price at time* $t-1$*,* P_{t-1}*, and using the percentage change formula:*

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100. \tag{2.4}$$

For instance, if ABC stock closes today at a price $P_t = \$11$ *while yesterday's closing price was* $P_{t-1} = \$10$*, then ABC stock's rate of return today is* 10%. *In other words, you made one dollar on the 10 dollars invested in this stock. The trouble with the rate of return on a stock is not its computation, rather it is its expectation. Fluctuations in stock prices could lead to a positive (gain) or negative (loss) rate of return and therein lies the uncertainty. For instance, if the price of ABC stock is* ***predicted*** *to increase, then the rate of return is expected to be positive and, consequently, one would expect to achieve a capital gain from holding that stock and vice-versa if the price of ABC stock is predicted to decrease. The prediction of whether the stock price is going to appreciate or depreciate in the future depends on the probabilistic future state of nature. If our intention is to compute the mean/average/expected rate of return, then depending on the given data, we can either compute the sample mean or the population mean. If a sample of observations on historical price data was collected, then we can use the sample mean formula and compute* $\overline{r}$ *as in (2.1). Alternatively, if we have data on the future states of nature and the corresponding probabilistic returns, then we can compute* Er *from the population mean formula in (2.3). Assume that the economy can be described by four states: steady state, expansion, recession, and recovery. Suppose you obtained the following predictions regarding ABC's probabilistic rates of return in each state of the economy:*

State of the Economy	r (%)	p
Steady	$r_1 = 6$	$p_1 = 1/4$
Expansion	$r_2 = 7$	$p_2 = 1/4$
Recession	$r_3 = 3$	$p_3 = 1/4$
Recovery	$r_4 = 4$	$p_4 = 1/4$

What is the expected (or most likely) rate of return that an investor could realize on this stock? It is easy to see that investing in ABC stock will most likely yield a rate of return of 5% *as*

$$\mu \overset{or}{=} Er = 6\left(\frac{1}{4}\right) + 7\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) = 5\%.$$

But, what about the ***risk*** *of investing in this stock? The risk can be measured by the* ***deviation*** *of each realization of the random variable* r *from its mean, i.e., its most likely value of* 5%*. For instance, if the economy is in a steady state,* $r = 6\%$ *and the deviation from the mean value is*

$$r - Er \overset{or}{=} r - \mu = 6\% - 5\% = 1\%.$$

Thus, the risk here is measured by the 1% *fluctuation above the mean. On the other hand, if the economy is in a recession, the deviation from the mean is* $3 - 5 = -2\%$*. Deviations from the mean could be positive or negative as shown in the fourth column of Table 2. This implies that summing all deviations to obtain a measure of dispersion or risk is not possible as the positive and negative deviations will cancel out and there wont be anything left to measure. Fortunately, we can overcome this problem by simply squaring all deviations as shown in the last column of Table 2. Squaring each deviation and multiplying it by its probability of occurrence yields a dispersion measure known as the* ***variance*** *of the distribution, which we will denote by* σ^2 *as follows:*

$$var(r) \overset{or}{=} \sigma^2 = 1\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) = 2.5.$$

Finally, we reverse the previous step of squaring each deviation by taking the square root of the variance. The new measure, which we will denote σ*, is known as the* ***standard deviation*** *of the distribution and it is our measure of risk. The standard deviation of the random variable* r *is, therefore,*

$$sd(r) \overset{or}{=} \sigma = \sqrt{2.5} \approx 1.6\%.$$

Thus, given the various states of the economy, investing in stock ABC will most likely yield an expected rate of return of 5% *with* 1.6% *risk of fluctuation above and below that expected rate of return.*

In the same way of making the notational distinction between the sample mean and the population, it is customary in statistics to denote the population variance by σ^2 and the sample variance by s^2. As demonstrated from the previous example, the standard deviation as a measure of dispersion captures the total financial risk of an asset. In general, however, the standard deviation of a random variable measures the deviation of its data points or observations from its central tendency.

Now, we are ready to give a formal definition of the variance (and the standard of deviation) of a random variable.

State of the Economy	r_i (%)	p_i	$r_i - \mu$	$(r_i - \mu)^2$
Steady	$r_1 = 6$	$p_1 = 1/4$	$r_1 - \mu = 6 - 5 = 1$	$(r_1 - \mu)^2 = 1$
Expansion	$r_2 = 7$	$p_2 = 1/4$	$r_2 - \mu = 7 - 5 = 2$	$(r_2 - \mu)^2 = 4$
Recession	$r_3 = 3$	$p_3 = 1/4$	$r_3 - \mu = 3 - 5 = -2$	$(r_3 - \mu)^2 = 4$
Recovery	$r_4 = 4$	$p_4 = 1/4$	$r_4 - \mu = 4 - 5 = -1$	$(r_4 - \mu)^2 = 1$
Mean: $Er = \mu = \sum r_i \times p_i = 5\%$ **Variance:** $var(r) = \sigma^2 = \sum (r_i - \mu)^2 \times p_i = 2.5$ **Standard Deviation:** $sd(r) = \sigma = \sqrt{\sigma^2} = \sqrt{2.5}$				

TABLE 2. Mean, variance, and standard deviation of the expected rate of return on ABC stock.

DEFINITION 15. *Let x be a random variable with* ***population mean*** $Ex = \mu$. *Let $x_1, ..., x_n$ be n realizations/observations on x corresponding to n states with probabilities $p_1, ..., p_n$. The* ***population variance*** *of a random variable x, denoted by σ^2, is the expected value of the square of the deviations from the mean, and is computed from the equation*

$$var\,(x) \overset{or}{=} \sigma^2 = (x_1 - \mu)^2 \times p_1 + (x_2 - \mu)^2 \times p_2 + \cdots + (x_n - \mu)^2 \times p_n, \tag{2.5}$$

The ***population standard deviation****, σ, is the square root of the population variance; that is,*

$$sd\,(x) = \sqrt{var\,(x)} \overset{or}{=} \sqrt{\sigma^2}. \tag{2.6}$$

DEFINITION 16. *Let x be a random variable and $x_1, ..., x_n$ be a sample of n realizations/observations on x. The* ***sample mean*** *is*

$$\overline{x} = \frac{x_1 + \cdots + x_n}{n}.$$

The ***sample variance*** *of x, denoted by s^2, is the weighted sum of squares of deviations from the sample mean, and is defined by the equation*

$$s^2 = \frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n}. \tag{2.7}$$

The ***sample standard deviation****, s, is the square root of the sample variance; that is,*

$$s = \sqrt{s^2} \tag{2.8}$$

REMARK 1. *In statistics, when calculating the sample variance from a small sample, i.e., a sample of size $n < 30$, we divide by $n - 1$ to account for the degrees of freedom. For large samples, however, $n - 1$ is very close to n, so we just divide by n. The sample variance formula in equation (2.7) is expressed under the presumption that the sample is large, i.e., $n > 30$. In practice, this is usually the case.*

Here is an example on how to compute the sample variance.

Example 8. *The first column of the table below gives 4 sample realizations pertaining to the random variable x. It is easy to see that the sample mean $\overline{x} = 25$. The second and third columns of the table show, respectively, the deviation and the square of deviation of each realization from the sample mean.*

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
10	$10 - 25 = -15$	225
30	$30 - 25 = 5$	25
40	$40 - 25 = 15$	225
20	$20 - 25 = -5$	25
$\sum x_i = 100$	$\sum (x_i - \overline{x}) = 0$	$\sum (x_i - \overline{x})^2 = 500$

Thus, the sample variance of the random variable x is

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n-1} = \frac{500}{3} = 166.7,$$

and the standard deviation is

$$s = \sqrt{s^2} = \sqrt{166.7} = 12.9.$$

Notice here, since the sample is small, we divided the sample variance by $n - 1$.

Problem 3. *Suppose you were given the following data on the amount spent on groceries for a random sample of customers at La Boulangerie deli store.*

10	23	30	39	58	70
13	24	32	39	58	70
15	25	34	40	58	74
15	25	35	41	58	80
20	25	37	48	63	82
21	25	38	53	64	88

Answer the following questions: (1) Define the observation unit and the random variable under consideration. What is the scale of this random variable? (2) Let the random variable be denoted by x. Compute the sample mean and the sample standard deviation of this random variable and explain their meanings.

Solution 3. *The observation unit is the deli. The random variable, x, is the amount spent on groceries by a typical customer. The scale is the dollar amount. The formula for the sample mean, $\overline{x}$, is found in equation (2.1) as*

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

where $n = 36$ is the sample size. Notice here that $n > 30$ and, thus, we have a large sample. To find the sample mean, we can use a spreadsheet to do the computations. This gives

$$\overline{x} = \frac{1530}{36} = 42.5.$$

The formula for the sample variance, denoted by s^2*, is found in equation (2.7) as follows:*

$$\begin{aligned} s^2 &= \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} \\ &= \frac{16359}{36} \\ &= 454.4 \end{aligned}$$

The standard deviation, s, is

$$s = \sqrt{s^2} = \sqrt{454.4} = 21.3,$$

which means that, on average, customers spend $42.5 on groceries with a standard deviation of $21.3 dollars, i.e., the variability around this average is +/- $21.3.

5. Risk: The Meaning of the Standard Deviation

Before we examine the association between two random variables or the so-called **covariance**, it is worth noting the difference between the variance and the standard deviation of a random variable. Make no mistake that it is the standard deviation that we are interested in calculating in order to measure the dispersion/deviation from the mean. Some elementary textbooks on the subject do not even mention the variance in the treatment of moments of the random variables. The variance, nevertheless, is important in deriving theoretical results for it is messier to work under the square root. Another significant difference between both concepts has to do with units of measurements. In contrast to $var(x)$, the quantities x, Ex, and $sd(x)$ are all comparable because they are all measured in the same units of measurement. For instance, if x is the length of a table in centimeters, then the mean and standard deviation of x are also measured in centimeters, and hence are comparable, whereas the variance is measured in centimeters square, which is a different unit of measurement. In financial applications, if x is the rate of return (in %) on an asset, then x, Ex, and $sd(x)$ are all measured in percentages. The variance, however, is measured in percentages square. The reader should keep in mind, however, that it is the standard deviation, as a measure of risk, that we are interested in.

The standard deviation of the rate of return on a financial asset r, $sd(r)$, whether computed from sample or population data, is a measure of the dispersion of returns around the average or the expected return. For instance, if r is the rate of return on a particular stock and if the expected rate of return on that stock $Er = 10\%$ with a standard deviation $sd(r) = \sigma = 2\%$, then the investor should expected 2% fluctuation either up or down around the required or expected rate of return of 10%. In other words, the risk or volatility of the returns pertaining to this asset is $+/-2\%$ above and below the 10% expectation. This is how total volatility or financial risk of an asset is precisely measured in practice.

PROBLEM 4. *Suppose you gathered a sample of monthly closing prices of XYZ stock over the period between June 2010 and November 2010. The data is shown in the table below, where* P_t *denote the stock price at time/month* t*. What is the required rate of return on XYZ? Explain the meaning of the resulting number. What is the risk of XYZ? Explain the meaning of the resulting number.*

$t =$ *month*	P_t ($)
June	10
July	12
August	15
September	12
October	15
November	18

SOLUTION 4. *In this problem, we have monthly historical data on XYZ stock. We need to compute the RRR on this stock using sample data. In other words, we need to find the sample average of returns on XYZ. This number is equivalent to the investor's expected or required return on that stock. Before we begin, we need, first, to convert prices into rates of return. Using*

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100\%$$

and

$$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n},$$

Table 3 shows how the rate of return at time t*, denoted by* r_t*, is derived from the price data. Notice that the first observation is lost since the price data on the month of May is not available. Thus, the workable sample number of observations for the rate of return* r_t *is* 5 *instead of* 6 *observations. Therefore, the sample mean rate of return on XYZ stock is*

$$\overline{r} = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{70}{5} = 14\%.$$

So, based on the historical data provided, the average rate of return on the stock is 14%*. This is the required or expected rate of return on XYZ. Now, to compute the risk of the stock, we ought to compute the* ***sample standard deviation*** *of the previous returns on XYZ. The* ***sample variance*** *of the expected rates of return on XYZ is computed as:*

$$var\,(r) = s_r^2 = \frac{\sum_{t=1}^{n} (r_t - \overline{r})^2}{n-1} = \frac{1470}{4} = 367.5,$$

where the quantity $\sum_{t=1}^{n} (r_t - \overline{r})^2 = 1470$ *is computed as shown from Table 3. As for the sample standard deviation, it is simply the square root of the*

$t =$ Month	$P_t\ (\$)$	$P_{t-1}\ (\$)$	$^{\ddagger}r_t\ (\%)$	$(r_t - \overline{r})$	$(r_t - \overline{r})^2$
June	10	NA†	NA†	NA†	NA†
July	12	10	20	$20 - 14 = 6$	36
August	15	12	25	$25 - 14 = 11$	121
September	12	15	-20	$-20 - 14 = -34$	1156
October	15	12	25	$25 - 14 = 11$	121
November	18	15	20	$20 - 14 = 6$	36
			$\sum_{t=1}^{n} r_t = 70$		$\sum_{t=1}^{n} (r_t - \overline{r})^2 = 1470$
$s^2 = \frac{\sum_{t=1}^{n}(r_t - \overline{r})^2}{n-1} = \frac{1470}{4} = 367.5.$					
†Not available $^{\ddagger}r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100\ (\%)$					

TABLE 3. The sample variance rate of return on XYZ stock over a period of 5 months.

sample variance as

$$sd\,[r] = s = \sqrt{s^2} = \sqrt{367.5} = 19.1.$$

This means that the most likely (or the expected) or required rate of return on XYZ is 14% with an expected volatility of +/-19.1% above and below this sample average.

PROBLEM 5. ***A simple two-period-three-states model.*** *Let today be period* 0 *and the future (or tomorrow) be period* 1. *Consider an investor in period* 0*, who is contemplating making the decision to invest in stock A. The economic conditions in period* 1 *are probabilistic and, thus, uncertain. Suppose the investor is expecting three states of nature:* $s_1, s_2,$ *and* s_3 *in period* 1*, with probabilities* $\frac{1}{2}, \frac{1}{4},$ *and* $\frac{1}{4}$ *respectively. Let* P_{A0} *and* P_{A1} *denote stock A's price in period* 0 *and period 1 respectively. Also, since period 1 depends on the state of nature, let* $(P_{A1})_{s_1}$, $(P_{A1})_{s_2}$, *and* $(P_{A1})_{s_3}$ *denote stock A's price in period 1 in states 1,2, and 3 respectively. The following data pertain to our two-period-three-states setup:*

$$P_{A0} = \$10$$

$$(P_{A1})_{s_1} = \$10,\ (P_{A1})_{s_2} = \$10.2,\ \textit{and}\ (P_{A1})_{s_3} = \$10.2.$$

Given the previous data, what is the investor's expected or required rate of return on stock A? Find a measure of volatility for stock A and explain its meaning.

SOLUTION 5. *The two-period model is a typical model that describes static buy-and-hold investment strategies in finance. In our case, stock A is traded today, i.e., in period 0, at a price* $P_{A0} = \$10$. *The future, or period 1, is* *probabilistic. It consists of three states* s_1, s_2, *and* s_3. *In each state, we have a predicted price; that is,* $(P_{A1})_{s_1} = \$10$ *if state* s_1 *occurs with probability* $\frac{1}{2}$, *or* $(P_{A1})_{s_2} = \$10.2$ *if state* s_2 *occurs with probability* $\frac{1}{4}$, *or*$(P_{A1})_{s_3} = \$10.2$ *if state* s_3 *occurs with probability* $\frac{1}{4}$. *Remember though that these are prices (and not returns). So, first, we need to find the predicted returns in each state in period 1 as follows:*

$$\begin{aligned}(r_A)_{s_1} &= \frac{(P_{A1})_{s_1} - P_{A0}}{P_{A0}} \times 100\% = \frac{10-10}{10} \times 100\% = 0\%,\\ (r_A)_{s_2} &= \frac{(P_{A1})_{s_2} - P_{A0}}{P_{A0}} \times 100\% = \frac{10.2-10}{10} \times 100\% = 2\%,\end{aligned}$$

and

$$(r_A)_{s_3} = \frac{(P_{A1})_{s_3} - P_{A0}}{P_{A0}} \times 100\% = \frac{10.2-10}{10} \times 100\% = 2\%.$$

Thus, now we have forecasted returns next period in three states with three probabilities:

State	*probability*	*return* (%)
s_1	$\frac{1}{2}$	0
s_2	$\frac{1}{4}$	2
s_3	$\frac{1}{4}$	2

Thus,

$$RRR = Er = \left(0 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{4}\right) = 1\%.$$

This is the required rate of return on stock A. As for σ_A, *we can first compute the variance as follows:*

$$\begin{aligned}var\,(r_A) \overset{or}{=} \sigma_A^2 &= \frac{1}{2} \times (0-1)^2 + \frac{1}{4} \times (2-1)^2 + \frac{1}{4} \times (2-1)^2\\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.\end{aligned}$$

Therefore,

$$sd\,(r_A) \overset{or}{=} \sigma_A = \sqrt{1} = 1.$$

Thus, the volatility of stock A is expected to be +/- 1% above and below the population mean of 1%.

Remark 2. *When computing the means (or expected returns) and variances (or risks), some would keep the returns in percentages, i.e., 1 as 1 percent, whereas others use two decimal places in expressing the percentages, i.e., 1 percent as* $\frac{1}{100} = 0.01$. *The two ways are correct. However, the former is preferred since one need not to keep track of four decimal places when computing the variance. Remember that the variance is measured in percentage* *square. For instance, in the previous problem, we could have stated period 1's returns as follows:*

State	*probability*	*return*
s_1	$\frac{1}{2}$	0.00
s_2	$\frac{1}{4}$	0.02
s_3	$\frac{1}{4}$	0.02

Then,

$$RRR = Er = \left(0 \times \frac{1}{2}\right) + \left(0.02 \times \frac{1}{4}\right) + \left(0.02 \times \frac{1}{4}\right) = 0.01.$$

Notice here that the percentage sign (%) was removed since all returns are already expressed in percentages. The variance is

$$\begin{aligned} var\left(r_A\right) \overset{or}{=} \sigma_A^2 &= \frac{1}{2} \times (0-0.01)^2 + \frac{1}{4} \times (0.02-0.01)^2 + \frac{1}{4} \times (0.02-0.01)^2 \\ &= 0.0001, \end{aligned}$$

and the standard deviation is

$$sd(r_A) = \sqrt{0.0001} = 0.01$$

or 1%. The reader should feel free to express the percentages as whole integers or using decimal places. It is paramount, however, to be consistent. Otherwise, the results would be misleading.

6. Covariance

The term covariance means the co-variability of two random variables. The intention is to measure the degree of association between two random variables, x and y say. The variability here means, as usual, the dispersion/deviation of the data from the mean; that is, we wish to measure $x - Ex$ and $y - Ey$ together. Here is a formal definition.

DEFINITION 17. *Consider n states of nature. Let $x_1, ..., x_n$ and $y_1, ..., y_n$ be the observations on x and y, respectively, in the n states. Let $p_1, ..., p_n$ be the state probabilities. Then, the* ***population covariance*** *of the two random variables x and y, $cov\,(x, y)$, is denoted by σ_{xy} and is the sum of all the deviations of each observation on x and y times the corresponding state probability; that is,*

$$\begin{aligned} cov\,(x,y) \overset{or}{=} \sigma_{xy} &= \left(x_1 - \mu_x\right)\left(y_1 - \mu_y\right) \times p_1 \\ &\quad + \left(x_2 - \mu_x\right)\left(y_2 - \mu_y\right) \times p_2 \\ &\quad \vdots \\ &\quad + \left(x_n - \mu_x\right)\left(y_n - \mu_y\right) \times p_n \end{aligned} \tag{2.9}$$

where $\mu_x = Ex$ and $\mu_y = Ey$ are the population means of x and y respectively.

For instance, if we have two states with two probabilities p_1 and p_2. Then,

$$\begin{aligned} cov\left(x,y\right) &= \left(x_1-\mu_x\right)\left(y_1-\mu_y\right)\times p_1 \\ &\quad +\left(x_2-\mu_x\right)\left(y_2-\mu_y\right)\times p_2. \end{aligned}$$

Notice that the covariance between x and itself is just the variance of x. To see this, just replace y by x in the previous equation. This gives:

$$\begin{aligned} cov\left(x,x\right) &= \left(x_1-\mu_x\right)\left(x_1-\mu_x\right)\times p_1 \\ &\quad +\left(x_2-\mu_x\right)\left(x_2-\mu_x\right)\times p_2 \\ &= \left(x_1-\mu_x\right)^2\times p_1+\left(x_2-\mu_x\right)^2\times p_2 \\ &= var(x). \end{aligned}$$

REMARK 3. *The covariance between a random variable and itself is just the variance of this random variable.*

DEFINITION 18. *Consider n observations, $x_1,...,x_n$ and $y_1,...,y_n$, on the random variables x and y, respectively. The* ***sample covariance*** *of the two random variables x and y, $cov\left(x,y\right)$, is denoted by s_{xy} and is the average of the sum of the product of the deviations from sample means; that is,,*

(2.10)

$$cov\left(x,y\right)\overset{or}{=}s_{xy}=\frac{\left(x_1-\overline{x}\right)\left(y_1-\overline{y}\right)+\left(x_2-\overline{x}\right)\left(y_2-\overline{y}\right)+\cdots+\left(x_n-\overline{x}\right)\left(y_n-\overline{y}\right)}{n},$$

where $\overline{x}$ and $\overline{y}$ are the sample means of x and y respectively, provided that the sample is large, i.e., $n>30$.[2]

Here is an example on how to compute the **population covariance** between two random variables.

EXAMPLE 9. *Consider two stocks A and B. Let r_A and r_B be the rates of returns on A and B respectively. Suppose your financial analyst gathered the following return data on both stocks in different states of the economy:*

State of the Economy	r_A	r_B	*State Probability*
Recession	-0.2	-0.4	0.2
Steady State	0.2	0.5	0.5
Expansion	0.5	0.3	0.3

Expected returns at every state and the corresponding probabilities

Measure the association between r_A and r_B, i.e., find the covariance between the returns on A and B. From the population covariance formula in (2.9), we need first to find the population means, or the expected rates of return on the two stocks. Using the expectation formula in (2.3), we have:

$$\begin{aligned} Er_A\overset{or}{=}\mu_A &= r_{A1}\times p_1+r_{A2}\times p_2+r_{A3}\times p_3 \\ &= (-0.2)(0.2)+(0.2)(0.5)+(0.5)(0.3)=0.21\ \textit{(or }21\%\textit{)} \end{aligned}$$

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[2] If the sample is small, i.e., $n<30$, we should, as usual, divide the numerator of equation (2.10) by $n-1$.

and

$$\begin{aligned} Er_B \overset{or}{=} \mu_B &= r_{B1} \times p_1 + r_{B2} \times p_2 + r_{B3} \times p_3 \\ &= (-0.4)(0.2) + (0.5)(0.5) + (0.3)(0.3) = 0.26 \text{ (or 26\%).} \end{aligned}$$

Now, we can compute the population variance from equation (2.9). The table below is useful to perform such lengthy computations.

State	p_i	r_{Ai}	r_{Bi}	$r_{Ai}-\mu_A$	$r_{Bi}-\mu_B$	$(r_{Ai}-\mu_A)(r_{Bi}-\mu_B)$
Recession	0.2	−0.2	−0.4	−0.41	−0.66	0.2706
Steady	0.5	0.2	0.5	−0.01	0.24	−0.0024
Expansion	0.3	0.5	0.3	0.29	0.04	0.0116

Thus,

$$\begin{aligned} cov(r_A, r_B) \overset{or}{=} \sigma_{AB} &= (r_{A1}-\mu_A)(r_{B1}-\mu_B) \times p_1 + (r_{A2}-\mu_A)(r_{B2}-\mu_B) \times p_2 \\ &\quad + (r_{A3}-\mu_A)(r_{B3}-\mu_B) \times p_3 \\ &= 0.2706 \times 0.2 + -0.0024 \times 0.5 + 0.0116 \times 0.3 \\ &= 0.0564 \end{aligned}$$

Thus, the covariance between the rate of return on asset A and the rate of return on asset B is 0.0564. This number is measured in percentages square and it only signifies the sign; that is, we can only infer that the two assets are positively associated. We will see shortly that there is a better way to measure the association between any two random variables; the so-called ***correlation coefficient****.*

Here is another example on how to compute the **sample covariance**.

Example 10. *Suppose you gathered a sample of monthly closing prices of A and B stocks over the period between June 2010 and November 2010. The data is shown in the table below, where P_{At} and P_{Bt} denote stock A and B's closing prices at time/month t respectively. Compute the sample covariance between the rate of return on stock A and the rate of return on stock B.*

$t = Month$	P_{At}	P_{Bt}
June	*10*	*8*
July	*12*	*6*
August	*15*	*10*
September	*12*	*12*
October	*15*	*14*
November	*18*	*12*

Notice here that we want to compute the covariance between the rates of return (and not the covariance between prices). Therefore, we need to find the rates of return on asset A and B first. Using the price change formula *(see equation (2.4), we begin by converting end-of-month stock prices into returns as follows:*

$t =$ *Month*	P_{At} (\$)	P_{At-1} (\$)	$r_{At} = \frac{P_{At}-P_{At-1}}{P_{At-1}} \times 100$ (%)
June	10	$NA^{\dagger}$	$NA^{\dagger}$
July	12	10	$\frac{12-10}{10} \times 100 = 20$
August	15	12	25
September	12	15	-20
October	15	12	25
November	18	15	20
			$\sum_{t=1}^{n} r_{At} = 70$
†*Not Available*			

$t =$ *Month*	P_{Bt} (\$)	P_{Bt-1} (\$)	$r_{Bt} = \frac{P_{Bt}-P_{Bt-1}}{P_{Bt-1}} \times 100$ (%)
June	8	$NA^{\dagger}$	$NA^{\dagger}$
July	6	8	-25
August	10	6	67
September	12	10	20
October	14	12	17
November	12	14	-14
			$\sum_{t=1}^{n} r_{Bt} = 65$
†*Not Available*			

Notice that because we are using the price lag in the price change formula, one observation is missing. In other words, when we compute returns from prices, the effective sample size is reduced by one. In our case, the effective sample size is $6 - 1 = 5$ *observations. Now, once we have the returns, we can compute the expected rate of return on each stock, which is simply the sample mean, as*

$$\bar{r}_A = \frac{\sum_{t=1}^{n} r_{At}}{n} = \frac{70}{5} = 14\%.$$

and

$$\bar{r}_A = \frac{\sum_{t=1}^{n} r_{Bt}}{n} = \frac{65}{5} = 13\%$$

for asset A and B respectively. Next, to compute the covariance between the rates of return on A and B, we need to find the deviations of the returns from their corresponding sample means. We compute those values in the *following table:*

$t =$ *Month*	r_{At} (%)	r_{Bt} (%)	$a = (r_{At} - \overline{r}_A)$	$b = (r_{Bt} - \overline{r}_B)$	ab
June	$NA^\dagger$	$NA^\dagger$	$NA^\dagger$	$NA^\dagger$	$NA^\dagger$
July	20	-25	$20 - 14 = 6$	$-25 - 13 = -38$	-228
August	25	67	$25 - 14 = 11$	$67 - 13 = 54$	594
September	-20	20	$-20 - 14 = -34$	$20 - 13 = 7$	-238
October	25	17	$25 - 14 = 11$	$17 - 13 = 4$	44
November	20	-14	$20 - 14 = 6$	$-14 - 13 = -27$	-162
$\sum$ *(sum)*			0	0	10

Notice that the sum of all deviations from the expected return must be zero (columns 4 and 5 in the above table). The sample covariance between the rates of return on assets A and B is the sum of the product of both deviations divided by the effective sample size minus one (since it is a small sample); that is,

$$cov\,(r_A, r_B) \overset{or}{=} s_{AB} = \frac{\sum (r_{At} - \overline{r}_A)(r_{Bt} - \overline{r}_B)}{n-1} = \frac{10}{4} = 2.5.$$

This means that there exists a positive association between the rates of return on A and B. Again, the number 2.5, doesn't really give us any idea about the magnitude and the strength of this association. A better measure, as we will explore in the next section, is the correlation coefficient.

7. Correlation

As noted in the previous section, the covariance of two random variables x and y measures the **association** between them. This association could be positive or negative and it could be strong or weak, i.e., the association is characterized by its **sign** (direction) and its **strength**. For instance, if x and y are taken to be rates of return on two stocks, and if $cov\,(x, y) = 0.005$, then we can deduce that there is a positive association amounting 0.005 between the returns of the two. We can't deduce, however, whether this positive association is strong or weak. We don't even know which covariance value is considered weak or strong. In short, the covariance as a measure of association is helpful in identifying the direction of the association, but it fails to convey its magnitude. This problem calls for another measure of association that can achieve both objectives. This measure is known as the **correlation**, or the **correlation coefficient** of x and y.

Definition 19. *The* ***population correlation*** *of two random variables x and y, $corr\,(x, y)$, is denoted by ρ_{xy} and is defined by the equation*

$$corr\,(x, y) \overset{or}{=} \rho_{xy} \overset{defn}{=} \frac{cov\,(x, y)}{sd\,(x)\,sd\,(y)} = \frac{\sigma_{x,y}}{\sigma_x \sigma_y}, \tag{2.11}$$

provided that var (x) *and var* (y) *are strictly positive. The* ***sample correlation*** *coefficient is denoted by* r_{xy}*, and is defined in the exact same way but using sample data; that is,*

$$corr\,(x,y) \overset{or}{=} r_{xy} \overset{defn}{=} \frac{s_{xy}}{s_x s_y}, \tag{2.12}$$

provided that var (x) *and var* (y) *are strictly positive.*

Notice that the variance of x and the variance of y can never be negative and, therefore, the denominator of equation (2.11) (or equation (2.12)) can never be negative. It can be zero, however, and for that reason the variances are required to be strictly positive. This implies that both the correlation coefficient and the covariance have the same sign. But, the correlation coefficient is a better measure of association than the covariance as, in addition to conveying the direction, it can also convey the strength of the correlation between any two random variables for it is a, **unit free,** number that takes any value between -1 and $+1$. To see how the correlation coefficient is a unit free measure, it suffices to observe that the units of the numerator and the denominator of (2.11) cancel out.

Proving the assertion that

$$-1 \leq corr\,(x,y) \leq 1 \tag{2.13}$$

is not a trivial exercise. We will, however, take this assertion as a fact and use it without a proof. Notice the importance of the fact that the correlation coefficient is bounded between -1 and 1. The confinement of the correlation coefficient in the closed interval $[-1,1]$ makes it easier to understand the magnitude of the association between x and y. For instance, if the correlation coefficient between the rates of return on two stocks x and y is -0.8, one can immediately conclude that the two returns are **strongly negatively** correlated. The more the correlation coefficient moves towards its boundaries, i.e., -1 and $+1$, the stronger is the correlation between the two random variables under consideration. A correlation coefficient of $+1$ implies that the two random variables are **perfectly positively correlated**; this is one extreme. The other extreme corresponds to a correlation coefficient of -1 and, in this case, the two random variables are said to be **perfectly negatively correlated**. If the correlation coefficient is zero (the mid point of the interval $[-1,+1]$), then there is **no relation** between the two random variables for their covariance must be zero. In other words, the two random variables are uncorrected. If the correlation coefficient takes any value in the interval of $(0,+1)$, then the two random variables are said to be positively (but not perfectly) correlated. The positive correlation is considered strong if it is closer to $+1$ and weak if it is closer to 0. Similarly, for the negative interval $(-1,0)$.

DEFINITION 20. *Two random variables x and y are said to be uncorrelated (independent) if their covariance is zero; that is,*

$$x \text{ and } y \text{ are independent} \quad \text{if and only if} \quad cov\,(x, y) = 0. \tag{2.14}$$

If $cov(x, y) = 0$, then it immediately follows that $corr(x, y) = 0$.

DEFINITION 21. *Two random variables x and y are said to be uncorrelated (independent) if their correlation coefficient is zero; that is,*

$$x \text{ and } y \text{ are independent} \quad \text{if and only if} \quad corr(x, y) = 0. \tag{2.15}$$

EXAMPLE 11. *In Example 10, we calculated the covariance between the two stocks to be 2.5 and we learned that this number only conveys the sign, but not the magnitude. Now, let's find the correlation between the returns on assets A and B. First, we need to compute the sample standard deviation for A and B, respectively, as*

$$\begin{aligned} var(r_A) \overset{or}{=} s_A^2 &= \frac{\sum (r_{At} - \overline{r}_A)^2}{n-1} \\ &= \frac{6^2 + 11^2 + (-34)^2 + 11^2 + 6^2}{5-1} \\ &= \frac{1470}{4} = 367.5 \\ &\Longrightarrow sd(r_A) = s_A = \sqrt{367.5} = 19.1, \end{aligned}$$

and

$$\begin{aligned} var(r_B) \overset{or}{=} s_B^2 &= \frac{\sum (r_{Bt} - \overline{r}_B)^2}{n-1} \\ &= \frac{(-38)^2 + 54^2 + 7^2 + 4^2 + (-27)^2}{5-1} \\ &= \frac{5154}{4} = 1288.5 \\ &\Longrightarrow sd(r_A) = s_B = \sqrt{1288.5} = 35.9, \end{aligned}$$

t = *Month*	r_{At} (%)	r_{Bt} (%)	$(r_{At} - \overline{r}_A)^2$	$(r_{Bt} - \overline{r}_B)^2$
June	$NA^\dagger$	$NA^\dagger$	$NA^\dagger$	$NA^\dagger$
July	20	−25	$(20 - 14)^2 = 36$	$(-25 - 13)^2 = 1444$
August	25	67	$(25 - 14)^2 = 121$	$(67 - 13)^2 = 2916$
September	−20	20	$(-20 - 14)^1 = 1156$	$(20 - 13)^2 = 49$
October	25	17	$(25 - 14)^2 = 121$	$(17 - 13)^2 = 16$
November	20	−14	$(20 - 14)^2 = 36$	$(-14 - 13)^2 = 729$
$\sum$ *(sum)*			1470	5154

Thus,

$$corr(r_A, r_B) = \frac{cov\,(r_A, r_B)}{sd(r_A) sd\,(r_B)} \overset{or}{=} \frac{s_{AB}}{s_A s_B} = \frac{2.5}{(19.1)\,(35.9)} \approx 0.004.$$

Thus, there exists an extremely weak positive correlation between the return on asset A and B. In fact, the correlation coefficient is very close to 0. So, one can conclude that the two stocks are not correlated. Observed here how the correlation coefficient conveys the magnitude of correlation. It allows us to conclude that the two stocks exhibit extremely weak positive correlation. For practical purposes, we can conclude that the two stocks are not correlated.

8. Financial Application: Constructing a Portfolio

In this section, we explore how can we use all the previous notions of risks and returns to construct a simple portfolio of two assets. Suppose that you saved a $1,000 and you decided to invest that sum of money in the financial market. In particular, you are interested in two stocks: AAA and BBB. Let r_{Ai} and r_{Bi} be state i's forecasted annual return on AAA and BBB stocks respectively. Assume further that the economy is characterized by three states: recession, steady state, and expansion. suppose that you gathered some data regarding the forecasted annual returns on these two stocks under each state of the economy with the corresponding probability of occurrence. The data is given in Table 4.

A quick look at the data, one can immediately conclude that stock AAA will perform better than BBB during recession and expansion, whereas the opposite is true in a steady state. Accordingly, you decided to use the $1000 to construct a portfolio p by allocating $600 on stock AAA and $400 on stock BBB. The question here is: What is the expected rate of return on this portfolio and what is its measure of risk (or volatility). In other words, you seek to measure Er_p and σ_p.

First, the rate of return on this portfolio is just the weighted average of the returns on the assets forming it; that is,

$$r_p = w_A r_A + w_B r_B, \tag{2.16}$$

where $w_A = \frac{600}{1000} = 0.6$ is the weight of asset AAA in the portfolio and $w_B = \frac{400}{1000} = 0.4$ is the weight of asset BBB. The expected rate of return on the portfolio, Er_p, is calculated in the same way; namely,

$$Er_p = w_A Er_A + w_B Er_B, \tag{2.17}$$

where $Er_A = \mu_A$ and $Er_B = \mu_B$ are the expected rates of return on AAA and BBB respectively. Thus, to find Erp, we ought to find Er_A and Er_B.

As for the variance of the portfolio p, σ_p^2, it can be shown that it is computed from the following formula:

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_{AB}, \tag{2.18}$$

where σ_A^2 is the variance of asset AAA's returns, σ_B^2 is the variance of asset BBB's returns, σ_{AB} is the covariance between the rate of return on asset

State i of the Economy	r_{Ai}(%)	r_{Bi}(%)	State Probability
$i =$ Recession	-20	-40	0.2
$i =$ Steady State	20	50	0.5
$i =$ Expansion	50	30	0.3
Asset returns at every state and the corresponding probabilities			

TABLE 4. Asset returns and their corresponding probabilities.

AAA and BBB, and everything else is the same as before.[3] Thus, to find the variance of the rate of return on the portfolio, we need the variances and covariance between the two assets forming it. Remember that we are doing all that to ultimately compute the total volatility or risk of this portfolio.

$$\sigma_p = \sqrt{\sigma_p^2}$$

This problem is an interesting application of means and variances of assets' returns. Notice first that, in this problem, the return data on both assets reflect the various states of the economy and, thus, it clearly represents the **population**. Therefore, the mean or expected rate of return on each asset is computed using the conventional population formula as follows:

$$\begin{aligned} Er_A \overset{or}{=} \mu_A &= \sum r_{Ai} \times p_i \\ &= (-20)(0.2) + (20)(0.5) + (50)(0.3) = 21\%, \end{aligned}$$

and

$$\begin{aligned} Er_B \overset{or}{=} \mu_B &= \sum r_{Bi} \times p_i \\ &= (-40)(0.2) + (50)(0.5) + (3)0(0.3) = 26\%. \end{aligned}$$

We are, however, interested in computing the expected rate of return on a portfolio p that consists of both assets. As noted above,

$$\begin{aligned} Er_p &= w_A Er_A + w_B Er_B \\ &= (0.6) \times 21\% + (0.4) \times 26\% \\ &= 23\%. \end{aligned}$$

In order to compute the variance of the portfolio, which is given in equation (2.18), we need to find the variance of each asset and the covariance between both. The **population variances** of the rates of return on assets AAA and BBB are computed, respectively, as follows:

$$\begin{aligned} \sigma_A^2 &= \sum (r_{Ai} - \mu_A)^2 \times p_i \\ &= (-20-21)^2(0.2) + (20-21)^2(0.5) + (50-21)^2(0.3) \\ &= 589 \end{aligned}$$

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[3] Proving the formulas in equations (2.17) and (2.18) is beyond the scope of this textbook. The focus here is on the application of these formulas in the case of a portfolio of two assets.

State	p_i	r_{Ai}	r_{Bi}	$r_{Ai}-\mu_A$	$r_{Bi}-\mu_B$	$(r_{Ai}-\mu_A)(r_{Bi}-\mu_B)$
Recession	0.2	-20	-40	-41	-66	2706
Steady	0.5	20	50	-1	24	-24
Expansion	0.3	50	30	29	4	116

TABLE 5. Computing the population covariance between the rates of return of stocks AAA and BBB.

and

$$\begin{aligned}\sigma_B^2 &= \sum (r_{Bi}-\mu_B)^2 \times p_i\\ &= (-40-26)^2(0.2)+(50-26)^2(0.5)+(30-26)^2(0.3)\\ &= 1164.\end{aligned}$$

The covariance between the two assets' returns is computed using the following equation

$$\sigma_{AB} = \sum (r_{Ai}-\mu_A)(r_{Bi}-\mu_B) \times p_i.$$

The computation of the term on the right-hand-side of the previous equation is done by multiplying the last column of Table 5 by the probability values in the second column. This gives

$$\sigma_{AB} = 2706(0.2) - 24(0.5) + 116(0.3) = 564$$

Thus,

$$\begin{aligned}\sigma_p^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}\\ &= (0.6)^2\,589 + (0.4)^2\,1164 + 2(0.6)(0.4)564\\ &= 669,\end{aligned}$$

and hence, the portfolio risk is

$$\sigma_p = \sqrt{669} \approx 26\%.$$

Thus, constructing this portfolio will most likely yield an expected rate of return of 23% with 26% fluctuation above and below that expected rate.

The previous simple problem shows how a simple portfolio is constructed from just two individual assets. In practice, however, a well diversified portfolio should consists of a number of assets; usually between 25-30 stocks. This is done to ensure wide exposure and to lower risks.

REMARK 4. *It is worth noting that, when we construct a portfolio of stocks that consists of* n *individual assets, every time one changes the weights of the individual assets in the portfolio, the resulting expected return and risk of the portfolio change accordingly. The following example illustrates this point.*

EXAMPLE 12. *In the above analysis, we were able to construct a portfolio p that consists of two assets A and B such that the weight of asset A,* $w_A = 0.6$, *and the weight of Asset B,* $w_B = 0.4$. *The resulting portfolio had an expected rate of return* $Er_p = 23\%$ *and risk* $\sigma_p = 26\%$. *Now, consider another investor who wants to construct portfolio l using both assets, but with different allocation. Suppose this investor wishes to invest 50% of his wealth in A and 50% in B. What is the effect of this allocation on the portfolio? In other words, would portfolio l have different expected return and risk than portfolio p? The answer is yes. Every time you change the allocation, the risk and return of your portfolio change accordingly. To see this, compute* Er_l *and* σ_l *one more time using* $w_A = w_B = 0.5$. *This gives*

$$\begin{aligned} Er_l &= w_A Er_A + w_B Er_B \\ &= (0.5) \times 21\% + (0.5) \times 26\% \\ &= 23.5\% \end{aligned}$$

and

$$\begin{aligned} \sigma_l^2 &= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_{AB} \\ &= (0.5)^2\, 589 + (0.5)^2\, 1188 + 2(0.5)(0.5)564 \\ &= 726.25 \\ \Longrightarrow &\quad \sigma_l = \sqrt{726.25} \approx 27\% \end{aligned}$$

Notice that the latter allocation yields a portfolio l with a higher expected return of 23.5% (as opposed to 23%) and a higher risk of 27% (as opposed to 26%).

Now, we have all what we need to understand and measure the expected rate of return, Er, and the risk, σ, of any financial asset and a portfolio of two assets. Our last quest in this chapter is to make use of these notions to understand the investor's behavior when engaging in a financial contract.

9. Attitudes Towards Risk

In general, finance theory seeks to study the **equilibrium** price of financial assets that are traded in financial markets. An equilibrium theory ought to have a demand side and a supply side of these assets. This stems from the fact that an equilibrium price is the result of the intersection between demand and supply in any market. In financial markets, however, the supply of financial assets is fixed. The demand for financial assets, on the other hand, is not. Among other factors, it varies according to investors' behaviors towards investment opportunities. These investment opportunities naturally involve financial **risk**. Thus, in order to better understand the demand for financial assets, which is crucial to our understanding of the equilibrium price of financial assets, we need to first understand how financial risk is defined and measured and how investors' attitudes towards risk are determined. We already discussed how risk is defined and measured in practice. It remains to discuss the various attitudes towards risk.

In general, humans have different attitudes towards risk and return. Some are **risk lovers**, some are **risk averse**, and others are **risk neutral**. When making investment decisions, risk averse individuals are those who are willing to take risk only if there is a possibility for higher return. The potential high returns is considered a compensation for the extra risk that those investors are about to take. Risk lovers, on the other hand, don't need to see a pattern of high returns in order to take on a risky investment. In fact, they derive satisfaction from making risky decisions. A risk neutral investor is an investor who is totally indifferent to risk when making an investment decision.

DEFINITION 22. ***An intuitive definition of risk-aversion***. *An investor is said to be risk-averse if s/he is willing to take risk only if there is an expectation for higher return in the future. In other word, a risk-averse investor will accept more* σ *only if* Er *is higher.*

Here is an example that illustrates the risk-averse behavior.

EXAMPLE 13. *Consider an investor who has a portfolio, p, of financial assets. The expected rate of return on the portfolio is* $Er_p = 10\%$ *and the risk of the portfolio is* $\sigma_p = 2\%$. *Suppose the investor is contemplating the possibility of investing in another portfolio, l. If the expected rate of return on the new portfolio* $Er_l = 14\%$ *and the risk of the new portfolio is* $\sigma_l = 3\%$. *What can you say about the investor's attitude towards risk if you learned that s/he decided to acquire portfolio l? It is easy to see here that portfolio l will yield more expected return of 4% for one percent increase in risk. Thus, from Definition 22, this investor is indeed a risk-averse investor.*

Definition 22 states that a risk-averse investor is one who would only take risk if there is a possibility of higher expected return. In practice, most investors are risk-averse. Does this mean that they all exhibit the same behavior? The answer is yes. Most investors would require more expected rate of return to bear more risk. But, they differ in their degrees of risk aversion. Some investors are more risk-averse than others. In other words, some investors require higher expected rate of return than others for the same risk. The following example illustrates the various degrees of risk-aversions.

EXAMPLE 14. *Consider two investors A and B. Each investor has an existing portfolio with an expected rate of return of 12% and risk of 1%. Both investors are facing the following investment opportunities:*

$Portfolio$	Er	σ
l	$Er_l = 15$	$\sigma_l = 3.5\%$
p	$Er_p = 17$	$\sigma_p = 3.5\%$

Notice that both investment opportunities, i.e., both portfolios have the same risk. But, portfolio p has higher expected rate of return. If Mr. A chooses portfolio l and Mr. B chooses portfolio p. What can you say about the risk attitude of each investor? First, it is easy to see that both investors are risk averse. This is true from Definition 22. Now, notice that Mr. A requires 3% (15%-12%) more return for 2.5% (3.5%-1%) increase in risk, whereas Mr. B requires 5% (17%-12%) increase in return for the same increase in risk. Thus, Mr. B is more risk averse than Mr. A for he needs more compensation for accepting the same amount of risk.

In Example 14, we saw how the increase in expected return per the same increase in risk can give us an idea about the degree of risk aversion. But, what if the increase in risk is not the same? For instance, what will happen if both investors in Example 14 are facing the following two investment opportunities?

$Portfolio$	Er	σ
l	$Er_l = 15$	$\sigma_l = 2.5\%$
p	$Er_p = 17$	$\sigma_p = 4.5\%$

Who is more risk averse than the other? To measure the degree of risk aversion in this case, we ought to find the increase in expected return per unit of risk, or the so-called **marginal expected return per unit of risk**.

Definition 23. *The marginal expected return per unit of risk is the increase in expected return as a result of taking an investment opportunity divided by the corresponding change in risk as measured by the change in standard deviation of the investment opportunity; that is,*

$$\textit{marginal expected return per unit of risk} \;=\; \frac{\textit{change in } Er}{\textit{change in } \sigma}$$

Thus, the marginal expected return per unit of risk pertaining to investment opportunity l is 2 percent per one unit of risk; that is,

$$\frac{15-12}{2.5-1} = \frac{3}{1.5} = 2,$$

whereas the marginal expected return per unit of risk pertaining to investment opportunity p is approximately 1.4 percent per unit of risk; that is,

$$\frac{17-12}{4.5-1} = \frac{5}{3.5} \approx 1.4.$$

Thus, Mr. A, who requires more expected return per unit of risk, is more risk averse than Mr. B.

Definition 24. *The degree of risk-aversion is measured by marginal expected return per unit of risk of the investment opportunity. The higher the marginal expected return per unit of risk, the more is the degree of risk-aversion.*

10. Practice Problems

(1) Suppose you are considering making an investment decision in two properties (real estates assets), r_1 and r_2. Your analyst provided the following data on the rates of return on your properties, and the associated states of nature of an economy. Assume five states such that state i for $i = 1, 2, ..., 5$, has a corresponding probability p_i. Let r_{1i} and r_{2i} be the predicted rates of return on asset 1 and 2, respectively, in state i.

State i	p_i	$r_{1i}(\%)$	$r_{2i}(\%)$
$i = 1$: Booming	0.2	18	0
$i = 2$: Expanding	0.2	5	-3
$i = 3$: Steady	0.2	12	15
$i = 4$: Recession	0.2	4	12
$i = 5$: Depression	0.2	6	1

Answer the following questions:

(a) Compute the expected rate of return of each individual asset.

(b) Compute the volatility/risk of each individual asset and explain its meaning.

(c) What is the difference between covariance and correlation as measures of association between two random variables in general? Compute the covariance and the correlation coefficient between assets 1 and 2 and explain their meanings.

(d) Suppose you decided to allocate 50% of your wealth in asset 1 and 50% of your wealth in asset 2. Let the resulting portfolio be denoted by p. Find the expected rate of return on your portfolio and compute its volatility using the formulas given in question (3) above; that is, use

$$Er_p = w_1 Er_1 + w_2 Er_2$$

and

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}}$$

to compute the expected rate of return and the risk of your portfolio respectively.

(e) What about this allocation: 80% in asset 1 and 20% in asset 2? If this investor is a risk averse investor, is this allocation better than the one in question 5? Explain why or why not? (Hint: compute Er_p and σ_p using the new weights and make your assessment based on the definition of risk-aversion).

(2) Consider the current data on the probabilistic price of AAA stock: Today's price is \$12 and next period's price could be \$18 or \$8 with equal probabilities. Answer the following questions:

(a) What is the expected rate of return on that stock?

   (b) What is the risk of that stock? Explain the meaning of this calculated risk.
   (c) Suppose the investor wishes to allocate his initial wealth of \$24 on a portfolio that consists of one unit of AAA stock and one unit of BBB stock. If BBB stock is also currently traded at \$12 and has an expected rate of return of 2 percent, variance of 900, and if the covariance between both stocks is 25. Find the expected rate of return on this portfolio and its risk.

(3) **The simple two-period-three-states model revisited**. Let today be period 0 and the future (or tomorrow) be period 1. Consider an investor in period 0, who is contemplating making the decision to invest in a portfolio that consists of two stocks: A and B. The economic conditions in period 1 are probabilistic and, thus, uncertain. Suppose the investor is expecting three states of nature: s_1, s_2, and s_3 in period 1, with probabilities $\frac{1}{2}, \frac{1}{4}$, and $\frac{1}{4}$ respectively. Let P_{A0} and P_{A1} denote stock A's price in period 0 and period 1 respectively. Also, since period 1 depends on the state of nature, let $(P_{A1})_{s_1}$, $(P_{A1})_{s_2}$, and $(P_{A1})_{s_3}$ denote stock A's price in period 1 in states 1,2, and 3 respectively. Similar notation is used to denote period 0 and period 1 prices of stock B. The following data pertain to our two-period-three-states setup:

$$P_{A0} = \$10,$$

$$(P_{A1})_{s_1} = \$10,\ \ (P_{A1})_{s_2} = \$10.2,\ \text{and}\ \ (P_{A1})_{s_3} = \$10.2,$$

$$P_{B0} = \$18,$$

and

$$(P_{B1})_{s_1} = \$18.54,\ \ (P_{B1})_{s_2} = \$18,\ \text{and}\ \ (P_{B1})_{s_3} = \$18.36.$$

Given the previous data, answer the following questions:

   (a) Compute the expected rate of return on stock A.
   (b) Compute the expected rate of return on stock B.
   (c) Compute the risk of sock A.
   (d) Compute the risk of stock B.
   (e) Compute the covariance between stock A's rate of return on B's rate of return.
   (f) Compute the correlation coefficient between stock A's rate of return and B's rate of return and explain its meaning.
   (g) Construct a portfolio p that consists of the two individual stocks A and B such that the weight of A, $w_A = \frac{5}{9}$, and the weight of B in the portfolio, $w_B = \frac{4}{9}$. Compute the expected rate of return on this portfolio and its measure of risk.

(4) Define what it means by risk-aversion. Consider tow investors: A and B. Both investors own a portfolio that has a rate of return of

10% and risk of 1%. Suppose both investors are faced with the following investment opportunities:

$Portfolio$	Er	σ
l	$Er_l = 15$	$\sigma_l = 3.5\%$
p	$Er_p = 20$	$\sigma_p = 6\%$

If investor A chooses portfolio l while investor B chooses p. Which investor is more risk-averse than the other? Explain your reasoning.

CHAPTER 3

Capital Asset Pricing Model

Capital Asset Pricing Model (CAPM) is a simple model that explains the risk-reward relationship for any individual asset or portfolio of assets. The model was originally developed independently by Treynor (1961) and Sharpe (1964). Then, it was further extended and clarified by Lintner (1965). The CAPM is considered an **equilibrium** theory of price formation in financial markets. The **equilibrium** that we are referring to here is the financial equilibrium in capital markets where the demand for financial assets (or securities) is equal to the supply. In fact, CAPM assumes that all asset prices clear the markets of **all** assets. The CAPM model, unfortunately, does not explain or derive this financial equilibrium; rather it assumes that it is the result of market **efficiency**. This hypothesis is known as the **efficient markets hypothesis** (EMH), which was formalized in (1970) by Eugene Fama. The axiom posits that capital markets are ideal in the sense that, at any point in time, security prices in **any** market fully reflect all available information in that particular market.

Recall, from Chapter 1, that capital markets allocate ownership of the economy's capital stock. The participants in the capital markets are firms and investors. Firms are after making production-investment decisions whereas investors are after making consumption-investment decisions. In order to raise capital, firms borrow by issuing securities that represent the firms' activities in the capital market. The money raised from issuing these financial securities are used in production. Investors, on the other hand, are engaged in making investment decisions by choosing among the available securities in the market, i.e., lending the corresponding issuing firms, in order to smooth their consumption. Like any other market, the outcome of capital market is the market price of each traded security. Since the quantities traded are fixed, i.e., the supply is fixed, fluctuations in the demand side is what causes the prices of securities to vary. If security prices at any time fully reflect all available market information, the market is said to be an **efficient market**.

DEFINITION 25. *The efficient markets hypothesis (EMH) posits that at any point in time, security prices* ***fully reflect*** *all market information. In other words, market participants have perfect information regarding the financial assets traded in the market at any point in time.*

The EMH implies that all market participants have the same information about all the securities traded in the market. In other words no participant has any information advantage over another. Under this **perfect information** assumption, it is likely that all market participants will have the same expectations about the risks and the returns of the financial assets that are traded in the market, and consequently all market participants should make the same decisions. This, in turn, rules out any opportunities to exploit profit. To see how a profit opportunity might arise in a financial market, consider, for instance, the case where an investor expects that the price of an asset will increase in the future, i.e., the current asset price is under-priced. If this investor formed his/her expectations based on information that was not available to all market participants, then, to make profit, this investor ought to buy the asset now and wait to sell it in the future at a higher price. Such profit opportunities, which arise when asset prices are under or over priced, are ruled out when markets are assumed to be efficient. In other words, if there is a possibility of making profit in the market, then this market is said to be **inefficient**.

In finance, in general, there are two schools of thought: one believe in the EMH and the other doesn't. The proponents of market inefficiencies, or the so-called **behavioral finance school**, argue that asset prices could be under or over priced as a result of many factors. One factor could be the existence of information advantages that some market participants might have over others. Another factor has to do with the fact that investors overreact to financial news, which, in turn, causes market fluctuations based on irrational expectations. Empirical evidence from behavioral finance and neuroscience shows that capital markets are not efficient all the time; they behave in an efficient way sometimes but they surely exhibit periods of irrational behavior due to the overreaction of market participants (especially retail traders) to various news and/or predictions of shocks.

At any rate, there is evidence to support both schools; financial markets are efficient sometimes and inefficient some other times. It is worth noting though that CAPM, as a model of price formation, is built on the premise of market efficiency. The analytical result of the model, which we will discuss shortly, is derived under the EMH. Although that might not be necessarily the case in practice, still the model is useful in giving investors the ideal risk-reward relation of a financial asset or a portfolio of assets. This relation could be taken as the benchmark against which inefficiency can be measured and quantified. We will discuss how this can be achieved shortly. As for now, we begin by discussing the key axioms based on which the CAPM is built.

1. The Axioms of CAPM

The CAPM model of Sharpe (1964) and Lintner (1965) is built on 4 key assumptions:

AXIOM 1. **Risk Attitude Axiom**: Investors are risk-averse.

AXIOM 2. $(Er - \sigma)$ **Axiom**: Investors care only about the mean and standard deviation of their one period investment return.

AXIOM 3. **Perfect Information.** Investors timely receive all relevant information pertaining to their investment decisions, and have complete agreement about the future distributions of asset returns.

AXIOM 4. **No transaction costs.**

AXIOM 5. **Equilibrium Axiom**: Borrowing and lending are done at the risk-free rate, which is the same for all investors and does not depend on the amount borrowed or lent.

Axiom 1 is sensible; most investors in financial markets are risk-averse investors who are willing to take risk, σ, only if there is a possibility of higher returns, i.e., only when there is high Er. Axiom 2 stems from 1. A risk-averse investor cares only about the mean, i.e., expected return, and standard deviation i.e., risk, of return.

Axiom 3 is about perfect information. All investors have complete agreement about the distributions of returns, i.e., all investors have complete agreement about the future states of the economy and the corresponding price probabilities of the traded assets. This means that investors will have the same expectations regarding the rates of returns on the traded assets. In other words, this axiom ensures that markets are **efficient**. It is simply one way of stating the EMH. The reason it is stated as a complete agreement of distribution of returns, rather than just saying that financial markets are efficient, is because when CAPM was proposed in 1964 and 1965 the EMH has not been formalized yet. It was Fama (1970) who proposed the EMH in 1970.

Axiom 4 assumes that investors can borrow or lend in the financial market at no cost. This assumption is imposed to simplify the model by ruling out broker fees and any other costs associated with financial transactions. This assumption is imposed for simplicity and doesn't affect the validity of the model.

Finally, Axiom 5 ensures that the market is at equilibrium by equating the borrowing and lending. When we assume that investors can borrow and lend at the same risk-free rate, then we are creating a market mechanism in which investors construct portfolios by mixing the risk-free asset with the market or any other combinations of assets traded in the market. The **market portfolio**, which we will denote by M, is the value-weighted portfolio of all traded assets. The weight of every risky asset i, w_i, in the market portfolio is

$$w_i = \frac{\text{market value of all outstanding units of the asset}}{\text{total market value of all risky assets}}.$$

Thus, when we assume that borrowing and lending are done via a risk-free asset, then investors are assumed to construct portfolios such that a portion of their wealth is invested in the risk-free asset and the remainder in the market (or any other portfolio of assets). Here is an example to illustrate this point.

EXAMPLE 15. *Consider a market of assets, e.g., the Dow Jones Industrial Average (DJIA). The DJIA index itself is the market portfolio, which is denoted by M. Let* r_M *be the rate of return on the market portfolio. Consider a risk-free asset, e.g., a government bond. Let* rf *be the risk-free rate. If borrowing and lending are allowed via the risk-free rate, then it is that rate that allows the trading. Basically, it is a way of mimicking a financial market. Investors can mix the risk-free with the market to construct a portfolio by either borrowing or lending via the risk-free rate. To see this, suppose you have a $1000 and you want to construct a portfolio by mixing the risk-free asset with the market portfolio. You can invest $600 of your wealth in the risk-free asset; that is, the weight of the risk-free asset in your portfolio is* $w_{rf} = \frac{600}{1000} = 0.6$. *The remainder 40% is allocated on the market portfolio; that is, the weight of the market portfolio is* $w_M = \frac{400}{1000} = 0.4$. *Let* p *be the constructed portfolio. Then, from equation (2.16), the rate of return on portfolio* p *is*

$$\begin{aligned} r_p &= w_{rf} rf + w_M r_M \\ &= (0.6) rf + (0.4) r_M. \end{aligned}$$

This is how a portfolio that mixes the risk-free asset with the market is constructed. To see how lending and borrowing could be done via the risk-free rate, consider the case of an aggressive investor, with an initial level of wealth $1000, who wishes to invest 150% of his wealth in the market portfolio, i.e., $w_M = 1.5$. *To this end, the investor needs to borrow $500 (50% of his wealth) at the risk-free asset and invest the amount borrowed plus all his wealth in the market portfolio. Thus, the weight of* rf *in this case is* $w_{rf} = -0.5$. *Notice that the weight of the risk-free asset is negative since the investor is borrowing. Notice also that the sum of both weights should always add to one. Thus, in this case,*

$$\begin{aligned} r_p &= w_{rf} rf + w_M r_M \\ &= (-0.5) rf + (1.5) r_M. \end{aligned}$$

Example 15 highlights the rationale of using a risk-free asset as a mechanism by which investors can borrow and lend in the financial market. This assumption is made to enable us to derive the analytical results of the CAPM. In practice, however, borrowing and lending is facilitated via market exchanges. But, since it is difficult to theoretically model this, it is conventional in finance theory to describe lending and borrowing via a risk-free asset. It is worth noting that this assumption doesn't in any way affect the analytical results of the model.

PROBLEM 6. *Jackie borrowed \$1000 at the risk-free rate of 6 percent. She invested the borrowed money and her own money of \$2500 in a market portfolio with an expected rate of return of 15 percent. What is the expected rate of return on the portfolio that Jackie constructed?*

SOLUTION 6. *Let the constructed portfolio be p. It consists of the market portfolio and the risk-free asset. The weight of the risk-free asset is* $w_{rf} = \frac{-1000}{2500} = -0.4$ *since Jackie is borrowing at the risk-free rate. Notice that the \$1000 has a negative sign since it is borrowed. Jackie will invest the \$1000 borrowed plus all her wealth of \$2500 in the market portfolio. Thus, the weight of the market in her portfolio is*

$$w_M = \frac{1000 + 2500}{2500} = 1.4.$$

In other words, Jackie is borrowing 40% of her wealth at the risk-free rate and investing 140% of her wealth in the market portfolio. Notice that the sum of the weight should always add to 1. The expected rate of return on the constructed portfolio is, therefore,

$$\begin{aligned} Er_p &= w_{rf} \times rf + w_M \times Er_M \\ &= -0.4 \times 6 + 1.4 \times 15 \\ &= 18.6\%. \end{aligned}$$

Now, since we understand the meaning and reasoning behind each assumption, we turn our attention to the equilibrium relation entailed by CAPM

2. The Rationale of CAPM

The objective of CAPM is to study the risk-reward relation pertaining to individual risky assets. William Sharpe (1964) suggested that there exists a linear relationship between any stock return and its risk. This is the statement of the CAPM which we now turn to.

CAPM is a financial equilibrium theory that is built on the previous four assumptions. CAPM is after finding the **expected rate of return,** Er, or the **required rate of return**, RRR, by investors on individual assets. Basically, CAPM answers the question: What is the expected or required rate of return on asset j say (say), i.e., what is Er_j or RRR_j that an investor would require if s/he decided to invest his or her money in that asset?

Before answering the previous question, it is worth noting that, from the discussion in Chapter 2, we already know how to compute Er_j or RRR_j on any asset. This could be done by either computing the **sample mean**, if we have historical data on returns, or by computing the **population mean**, if we have data on future states and their corresponding probabilities. CAPM provides a **third** way to compute expected or required rates of return.

The second point to note is that, from the Axioms of CAPM, we ought to combine the risk-free asset with any other assets or portfolio of assets in order to mimic the market exchange. It is worth noting that the risk-free rate of return is the rate of return realized on a risk-less investment, e.g., government bonds. In other words, $\sigma = 0$ for any risk-free asset. Thus, the rate of return on any risk-free asset, rf, is considered the **minimum** rate of return that any investor could achieve without taking any risk. Therefore, in order to invest in a risky asset j, the expected/required rate of return on this asset, Er_j, has to be **higher** than the risk-free rate; otherwise, the investor will not be motivated to acquire the asset. Thus, the expected rate of return on asset j is simply the risk-free rate plus a **premium** for the risk that the investor has to tolerate as a result of buying the asset. The equation of the CAPM is, therefore,

$$Er_j = rf + \text{ Asset } j\text{'s risk premium.}$$

The next question is: How to compute Asset j's risk premium? Notice that asset j is one asset among many others that are traded in a financial market. Each time an asset is added to this market, it contributes to the total market risk, which is the total market volatility σ_M. Thus, asset j's risk premium is found by multiplying the **amount of risk** due to the asset itself (asset j's risk contribution) times a **general price of risk**.

$$\begin{aligned} Er_j &= rf + \text{ Asset } j\text{'s risk premium} \\ &= rf + \text{ amount of risk due to asset } j \times \text{general price of risk.} \end{aligned} \tag{3.1}$$

The amount of risk due to asset j is the risk contribution of asset j times the market risk σ_M. For instance, if asset j contributes a fraction equals to 0.2 of the total market risk, then the amount of risk due to asset j is $0.2 \times \sigma_M$. The risk contribution here is **not** the weight of the asset in the portfolio; rather it is just a coefficient known as the **beta coefficient** of asset j, and is denoted by $\beta_{j,M}$. Thus,

$$\text{amount of risk due to asset } j = \beta_{j,M}\sigma_M. \tag{3.2}$$

The beta coefficient of any asset measures how much this particular asset contributes to the market index/portfolio, i.e., the portfolio of all assets traded in the market. It is defined as the ratio of the association between that asset and the market portfolio over the market variance.

DEFINITION 26. *Let M be the market portfolio. The beta coefficient of an asset j, denoted by $\beta_{j,M}$, is the association between the rate of return on asset j and the market rate of return divided by the variability of the market rate of return; that is,*

$$\beta_{j,M} = \frac{cov\,(r_j, r_M)}{var\,(r_M)} = \frac{\sigma_{jM}}{\sigma_M^2}. \tag{3.3}$$

It is simply a measure of the risk contribution of asset j to the market.

Equation (3.2) states that asset j's risk contribution is equal to the asset beta coefficient times the market risk. The beta coefficient of asset j is the covariance risk of asset j in M measured relative to the variance of the market return. In economic terms, $\beta_{j,M}$ is the proportional to the risk each dollar invested in asset j contributes to the market portfolio. Thus, the beta coefficient of asset j represents its risk that can not be avoided, or in other words, its **systematic risk.**

DEFINITION 27. *The beta coefficient of an asset represents its systematic risk. It is the proportional to the risk each dollar invested in the asset contributes to the market portfolio.*

Definitions 26 states that the beta coefficient of an asset measures the risk contribution of the asset to the market. Take any market; the DJIA (say), and ask yourself: What is the risk contribution of this market index to itself? In other words, what is the beta of the DJIA? Well, if we let M denote the DJIA, then, from equation (3.3), we have

$$\beta_{M,M} = \frac{cov\,(r_M, r_M)}{var\,(r_M)}.$$

But, from Remark 3 on page 38 in Chapter 2, we know that $cov\,(r_M, r_M) = var(r_M)$. Thus,

$$\beta_{M,M} = \frac{cov\,(r_M, r_M)}{var\,(r_M)} = \frac{var\,(r_M)}{var\,(r_M)} = 1.$$

This means that the risk contribution of the market in itself is 1. Applying Definition 27, $\beta_{M,M} = 1$ means that each dollar invested in the market contributes 1 unit of risk to the market. This is the benchmark measure against which we can assess whether an asset contributes less or more to the total market volatility. If the beta of an asset is greater than 1, it contributes more to the market volatility. If it is less than one, it contributes less. For instance, consider the stock of Proctor and Gamble,– an asset traded in the DJIA. If the beta coefficient of this stock is 0.4, than this means that this particular stock contributes 0.4 to the DJIA. In other words, each dollar invested in this stock contributes 0.4 unit of risk to the DJIA.

REMARK 5. *Notice that the beta coefficient is* ***unit free****. This is easy to see since the unit of measurements of the numerator and the denominator in the definition of beta cancel out. So, a beta of coefficient of 0.4 doesn't mean 40%. Rather, it means, as explained above, that each dollar invested in this asset contributes 0.4 unit of risk to the market volatility.*

REMARK 6. *Stocks with beta coefficients higher than one are considered riskier than stocks with beta coefficients less than 1. This is because the former contributes more to the total market volatility.*

EXAMPLE 16. ***On the definition of the beta coefficient.*** *Consider the CAPM framework. Assume that investors can borrow/lend at the risk-free rate. Suppose an investor forecasted the correlation coefficient between the rate of return on RRU Mutual Fund, R, and the market portfolio, M, to be* $corr_{RM} = 0.8$*. The investor's forecast of the standard deviations of the rates of returns are* $\sigma_R = 0.25$ *and* $\sigma_M = 0.10$ *for R and M, respectively. Find the beta coefficient of the R fund. Recall the definition of the correlation between R and M:*

$$corr_{RM} = \frac{cov\left(r_R, r_M\right)}{\sigma_R \sigma_M}.$$

Since we have $corr_{RM}, \sigma_R,$ *and* σ_M*, then we can compute the covariance between the market return and the fund's return as follows:*

$$\begin{aligned} corr_{RM} &= \frac{cov\left(r_R, r_M\right)}{\sigma_R \sigma_M} \Longrightarrow 0.8 = \frac{cov\left(r_R, r_M\right)}{0.25 \times 0.10} \\ &\Longrightarrow cov\left(r_R, r_M\right) = 0.02. \end{aligned}$$

Now, since we have the covariance, we can find the beta coefficient of the R fund as follows:

$$\beta_{R,M} = \frac{cov\left(r_R, r_M\right)}{var\left(r_M\right)} = \frac{0.02}{(0.10)^2} = 2.$$

This means that for every dollar invested in mutual fund R contributes 2 units of risk to the market volatility.

The reader, at this point, might be wondering about the difference between β of an asset and σ? Both are measures of risk. The former, however, captures only the systematic part of it. The latter is the total risk (or total volatility). The idea here is that, the total risk (or total volatility), σ, associated with the return on any asset can be decomposed into two components: **systematic risk** and **non-systematic risk**. The former component is the beta coefficient of the asset. It is that component that the investor needs to be compensated for if he or she decided to hold that particular asset. This component of risk is also known as the **non-diversifiable** risk component for it can **not** be avoided by including more assets in a portfolio. The latter component (the non-systematic risk) is that part of risk that can be avoided by diversification, i.e., adding more assets in the portfolio and, therefore, it is also known as the **diversifiable** risk component. Thus,

σ_j = systematic/non-diversifiable risk + non-systematic/diversifiable risk,

where

$$\beta_{jM} = \text{systematic/non-diversifiable risk of asset } j.$$

It can be shown that investors can avoid the non-systematic risk completely by **diversification**, i.e., by adding more assets in their portfolios. The systematic part, however, is non-diversifiable and cannot be avoided by diversification. An example of systematic risk is the risk of GDP fluctuations, unemployment, inflation, political instability, or any other macro factor that might affect financial markets. These types of risk can not be eliminated by diversification.

The second component of equation (3.1) is the general price of risk, which is the **market premium**, i.e., the market rate of return in excess of the risk-free rate $(Er_M - rf)$, relative to the market risk σ_M; that is,

$$\text{price of risk} = \frac{(Er_M - rf)}{\sigma_M}, \tag{3.4}$$

which is also known as the **Sharpe ratio**.

DEFINITION 28. *The Sharpe ratio is the ratio of the market excess return over the market volatility. It is defined by the equation*

$$\textit{Sharpe Ratio} = \frac{(Er_M - rf)}{\sigma_M}. \tag{3.5}$$

It is simply a general price of risk pertaining to a particular market.

Substituting (3.2) and (3.4) back in (3.1), we get the general equation of the CAPM as follows:

$$Er_j = rf + \underbrace{\underbrace{\beta_{j,M}\sigma_M}_{\text{risk due to asset j}} \times \underbrace{\frac{(Er_M - rf)}{\sigma_M}}_{\text{price of risk}}}_{\text{risk premium}}, \tag{3.6}$$

which, with the minor cancellation of σ_M in the numerator and denominator of the two terms on the right-hand-side, yields

$$Er_j = rf + \underbrace{\beta_{j,M} \underbrace{(Er_M - rf)}_{\text{market premium}}}_{\text{asset } j\text{'s risk premium}}, \tag{3.7}$$

or equivalently,

$$(Er_j - rf\ ) = \underbrace{\beta_{j,M} \underbrace{(Er_M - rf)}_{\text{market premium}}}_{\text{asset } j\text{'s risk premium}}, \tag{3.8}$$

Equation (3.7) states that the expected/required rate of return on any risky asset j is the risk-free plus a premium that compensates the investor for holding such an asset. This premium is simply asset j's risk contribution in the market times a **fixed** market premium. Equation (3.8) states that asset j's premium, or the required rate of return on asset j in excess of the risk-free rate, $(Er_j - rf)$, is linearly related to the **market premium,** $(Er_M - rf)$, through the **systematic risk** of the asset, $\beta_{j,M}$.

Before we turn to a graphical illustration of this linear relation, it is constructive to illustrate the previous notions by means of some examples.

EXAMPLE 17. *Consider a risky stock A that is currently traded in a financial market. Let* r_A *be the rate of return on stock A and* r_M *be the market rate of return or the market index rate of return. Let* rf *be the risk-free rate, i.e., the rate of return on a government bond. Suppose you had historical data on* r_M *and you were able to compute the average rate of return, i.e., the expected rate of return of the market* Er_M*. If* $Er_M = 8\%$ *and* $rf = 2\%$*. Then this particular market gives a market premium of* $Er_M - rf = 6\%$*. That is, investing in this market, i.e., just holding the market index yields 6% above the minimum risk-free rate. That is the* ***fixed*** *premium that any investor would get if he or she invests in such a market. If the risk contribution (or the systematic risk) of this stock in the market is* $\beta_{AM} = 0.5$*, what is the expected or required rate of return for this investor from CAPM model? The investor would require a rate of return of 2% (risk-free rate) plus stock A's risk premium. Therefore, from CAPM equation, we can compute* Er_A *as follows:*

$$\begin{aligned} Er_A &= rf + \beta_{AM}(Er_M - rf) \\ &= 2\% + 0.5(6\%) \\ &= 5\%. \end{aligned}$$

In other words, the investor needs a least 5% to be willing to invest in stock A. That is the expected or required rate of return on this stock from CAPM.

We argued in Chapter 2 that the expected/required rate of return, Er, can be found from **sample data** or from **population data** (using the rule of expectation). As demonstrated in Example 17, CAPM provides an alternative way of computing Er. The reader should keep in mind that this way of computing Er is built on the four Axioms of CAPM. In other words, computing Er via CAPM implicitly implies that financial markets are **efficient**.

EXAMPLE 18. *Suppose the assumptions of CAPM hold. The risk-free rate is 3.5 percent, the market risk premium is 6 percent, and stock A has an expected return of 15.5 percent. What is the beta of stock A? In this example, the reader should pay attention to the meaning of the market risk premium.*

In the CAPM equation,

$$Er_i = rf + \beta_{i,M}\left(Er_M - rf\right),$$

the market risk premium is $(Er_M - rf)$. *In this example, we have* $Er_A = 15.5\%$, $rf = 3.5\%$, *and* $(Er_M - rf) = 6$. *Thus, direct substitution yields:*

$$\begin{aligned} 15.5 &= 3.5 + 6\beta_{A,M} \\ &\Longrightarrow \beta_{A,M} = 2. \end{aligned}$$

EXAMPLE 19. *Assume that the axioms of CAPM hold. Consider a risky stock A that is currently traded in a financial market. Let* r_A *be the rate of return on stock A and* r_M *be the market rate of return. Let* $rf = 2\%$ *be the risk-free rate. Your analyst was able to provide you with the following information regarding that market:* $Er_M = 8$, $\sigma_M = 2$, $cov\left(r_A, r_M\right) = 1$. *How can you use this data to compute your required/expected rate of return on stock A should you wish to invest in such asset? First, we compute the beta coefficient of stock A as follows:*

$$\beta_{A,M} = \frac{cov\left(r_A, r_M\right)}{var\left(r_M\right)} = \frac{1}{4} = 0.25.$$

Then, using CAPM equation, we have

$$\begin{aligned} Er_A &= rf + \beta_{AM}\left(Er_M - rf\right) \\ &= 2\% + 0.25(8\% - 2\%) \\ &= 3.5\%. \end{aligned}$$

In other words, the investor needs a least 3.5% to be willing to invest in stock A. That is the expected or required rate of return on this stock from CAPM.

PROBLEM 7. *Suppose you have the following historical return data on the market index and a stock A, which is traded in it.*

Year	$r_A(\%)$	$r_M(\%)$
1	+10	+12
2	−2	−1
3	+4	+10
4	−2	−2
5	+5	+1

Suppose that the CAPM assumptions hold. If the risk-free rate is 1%, compute the expected rate of return on Asset A. Find the Sharpe ratio and explain the meaning of your finding.

SOLUTION 7. *In this problem, we have the risk-free rate,* $rf = 1\%$, *and we can compute the expected rate of return on the market from the historical/sample data given as follows::*

$$\begin{aligned} Er_M &= \bar{r}_M = \frac{\sum r_{Mi}}{n} \\ &= \frac{+12 - 1 + 10 - 2 + 1}{5} = 4\% \end{aligned}$$

In order to find Er_A from the CAPM equation:

$$Er_A = rf + \beta_{A,M}\left(Er_M - rf\right),$$

we need to find the beta coefficient of asset A, which, by definition, is given by the equation

$$\beta_{A,M} = \frac{cov\left(r_A, r_M\right)}{var\left(r_M\right)}.$$

We proceed by constructing the following table:

$Year$	$r_A(\%)$	$r_M(\%)$	$a = (r_A - \overline{r}_A)$	$b = (r_M - \overline{r}_M)$	ab	b^2
1	$+10$	$+12$	$10 - 3 = 7$	$12 - 4 = 8$	56	64
2	-2	-1	$-2 - 3 = -5$	$-1 - 4 = -5$	25	25
3	$+4$	$+10$	$4 - 3 = 1$	$10 - 4 = 6$	6	36
4	-2	-2	$-2 - 3 = -5$	$-2 - 4 = -6$	30	36
5	$+5$	$+1$	$5 - 3 = 2$	$= 1 - 4 = -3$	-6	9

$$\overline{r}_A = \frac{\sum r_{Ai}}{n} = \frac{+10 - 2 + 4 - 2 + 5}{5} = 3\%$$

$$\begin{aligned} cov\left(r_A, r_M\right) &= \frac{\sum\left(r_A - \overline{r}_A\right)\left(r_M - \overline{r}_M\right)}{n} = \frac{111}{5}, \\ var\left(r_M\right) &= \frac{\sum\left(r_M - \overline{r}_M\right)^2}{n} = \frac{170}{5}, \end{aligned}$$

$$\beta_{A,M} = \frac{cov\left(r_A, r_M\right)}{var\left(r_M\right)} = \frac{\frac{111}{5}}{\frac{170}{5}} = 0.65$$

Thus,

$$\begin{aligned} Er_A &= rf + \beta_{A,M}\left(Er_M - rf\right) \\ &= 1 + 0.65(4 - 1) \\ &= 2.95\%. \end{aligned}$$

The Sharpe ratio is the ratio of the market excess return over the market volatility; that is,

$$\begin{aligned} Sharpe\ Ratio &= \frac{\left(Er_M - rf\right)}{\sigma_M} \\ &= \frac{4 - 1}{\sqrt{\frac{170}{5}}} = \frac{3}{5.8} = 0.5. \end{aligned}$$

It means that the excess return per unit of market risk is 0.5%. That is, for every increase in σ_M by 1 unit, the market will generate an excess return of 0.5%.

3. The Security Market Line (SML)

The CAPM equation in (3.7) is graphically illustrated in a two-dimensional $(Er - \beta)$ space, i.e., expected return-systematic risk space, by a straight line as shown in Figure 2. This line is known as the **security market line** (SML), where the dependent variable (the one on the $y - axis$) is Er and the independent variable (the one on the $x - axis$) is β. A quick look at the CAPM equation for a risky asset j;

$$Er_j = rf + \beta_{jM}(Er_M - rf),$$

one can notice that the SML is a line with an intercept equal to the risk-free rate and a slope equal to the **market premium** $(Er_M - rf)$. Notice that if we take asset j's return to be the market's return, r_M, the right-hand-side of the CAPM will agree with the left-hand-side since the **market beta** is 1 as shown from point M on the SML in Figure 2. Notice also that if we take asset j to be the risk-free asset, its beta coefficient is zero and, thus, its expected rate of return is the risk-free rate as shown from point $(0, rf)$ in Figure 2.

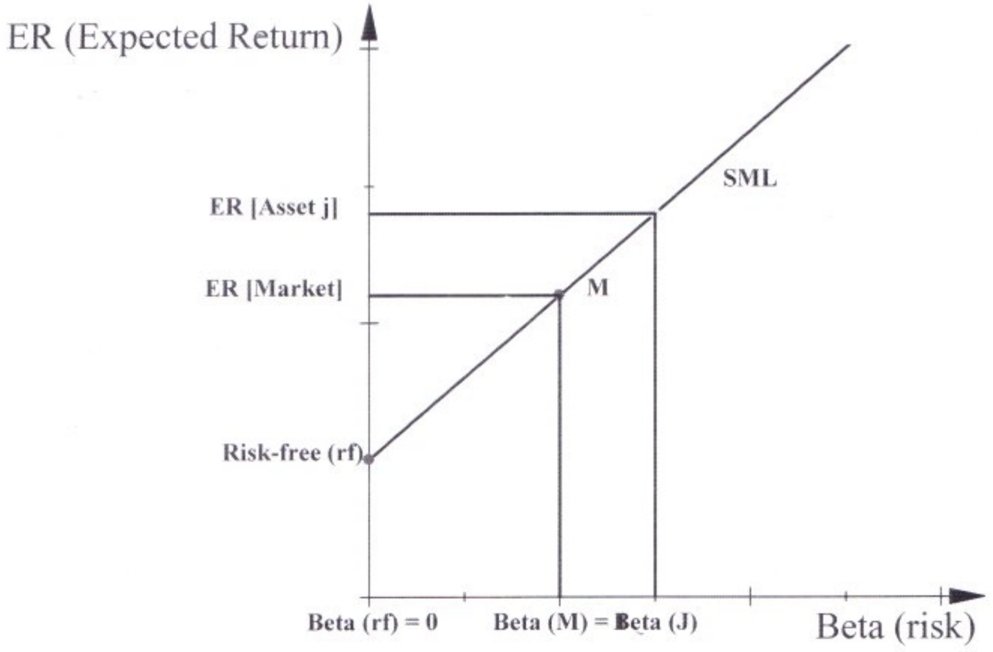

Figure 2: The Security Market Line (SML) shows the relation between the required rate of return on an asset and its market beta (systematic risk).

EXAMPLE 20. *Using monthly data on the rates of return on two stocks of two corporations: Apple (A) and Gillette (G), suppose you managed to estimate the beta coefficients for both stocks, respectively, as* $\beta_A = 1.4$ *and* $\beta_G = 0.6$. *Market data shows that the risk-free rate is* 5% *and the market premium is* 6%*; that is,* $rf = 5\%$ *and* $(Er_M - rf) = 6\%$. *What is the rate of return required by any investor to hold any of the two stocks? To find the required rate of return on Apple stock, we simply substitute the information on the* *risk-free rate and market risk premium along with our estimate of beta for Apple stock in the CAPM. This gives an expected (or required) rate of return of* 13.4% *as*

$$\begin{aligned} Er_{Apple} &= rf \; + \; \beta_{A,M}\,(Er_M - rf) \\ &= 0.05 + 1.4(0.06) = 13.4\% \end{aligned}$$

as shown from point A on the SML in Figure 3. The required rate of return on Gillette stock is computed in the same manner as

$$\begin{aligned} Er_{Gillette} &= rf \; + \; \beta_{G,M}\,(Er_M - rf) \\ &= 0.05 + 0.6(0.06) = 8.6\%, \end{aligned}$$

and the risk-reward relation is plotted in Figure 4.

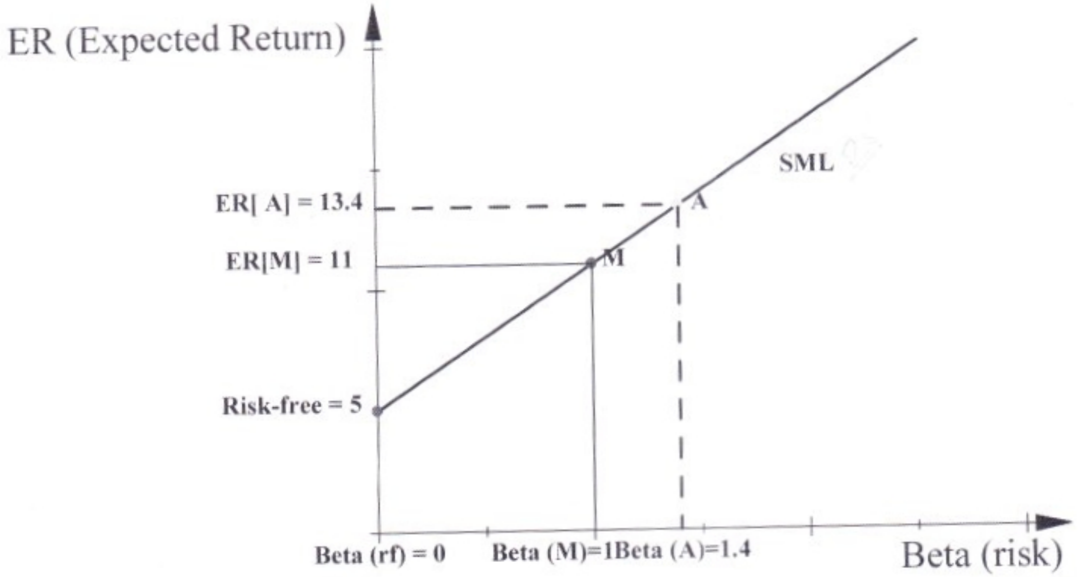

Figure 3: The SML showing the required rate of return (%) on Apple stock when the risk-free rate is 5%, the market premium is 6%, and the beta coefficient is 1.4.

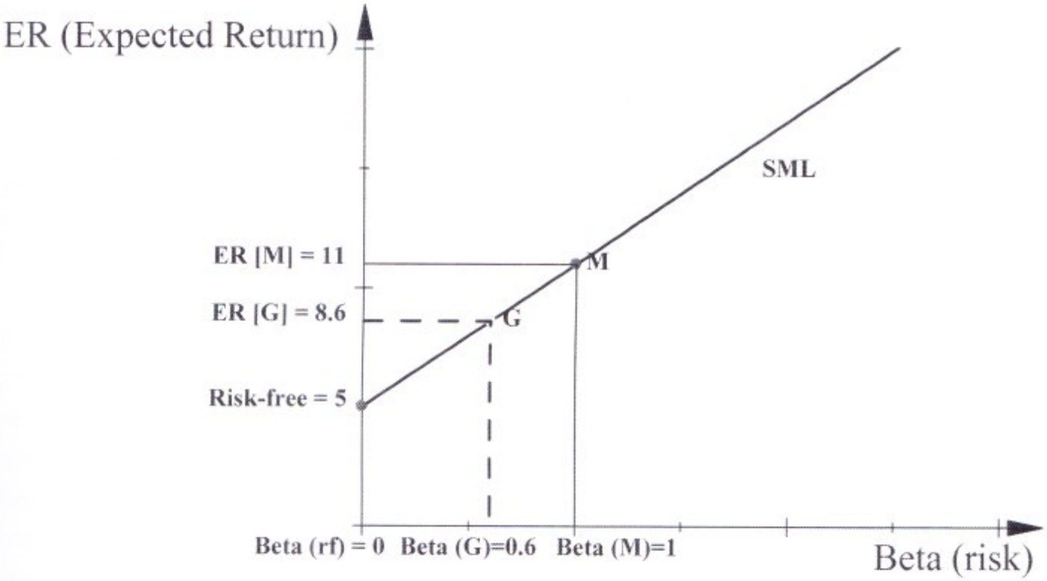

Figure 4: The SML showing the required rate of return (%) on Gillette stock when the risk-free rate is 5%, the market premium is 6%, and the beta coefficient is 0.6.

In example 20, we have shown how the risk-reward relation (in $Er-\beta$ space) pertaining to two stocks (Apple and Gillette) is depicted on the SML. The key point here is that, if all CAPM assumptions hold, the expected rate of return of any asset traded in the market must fall on the SML, e.g., points G and M. In other words, points off the SML are impossible if CAPM assumptions hold. Why this is the case? Why for a beta coefficient of 0.6, Gillette can never achieve an expected rate of return above 8.6% (see Figure 4)? The reason has to do with the axioms of CAPM. In particular, Axioms 3 and 5. Axiom 3 states that all information all available to market participants and, consequently, all participants are making the same expectations regarding Er and β of the assets. This means that, under perfect information, all market participants will make the same computations regarding Er and β of all assets traded in the market. Then, if we further assume that borrowing is equal to lending, then every asset's expected return and beta combination should lie on the SML. In other words, the SML represents the equilibrium relation in $Er-\beta$ space of every risky asset traded in the market if the assumptions of CAPM hold.

DEFINITION 29. *The SML is the graphical representation of the CAPM equation. It represents the equilibrium risk-reward relation in* $Er-\beta$ *space for every individual risky asset in the market.*

The following problem illustrates how the perfect information and equilibrium assumptions imply that all assets should obey the SML.

PROBLEM 8. *Stock A has a beta of 1.8 and an expected return of 20 percent. Stock B has a beta of 1.2 and an expected return of 14 percent. If CAPM holds, what should the return on the market and the risk-free rate be?*

SOLUTION 8. *This is a classic problem that illustrates the power of the axioms behind CAPM. It uses the fact that, if all CAPM assumptions hold, all stocks should be on the SML. Thus, stock A and B should satisfy the same CAPM equation. In particular,*

$$Er_A = rf + \beta_A(Er_M - rf)$$

holds for A and

$$Er_B = rf + \beta_B(Er_M - rf)$$

holds for B. Now, substitute the data given in both equations. This yields:

$$\begin{aligned} 20 &= rf + 1.8(Er_M - rf) \text{ and} \\ 14 &= rf + 1.2(Er_M - rf). \end{aligned}$$

You get two equations in two unknowns rf and $(Er_M - rf)$. To solve this, isolate $(Er_M - rf)$ from the first equation and substitute in the second; that is, from the first equation, obtain

$$(Er_M - rf) = \frac{20 - rf}{1.8}.$$

Now, plug this in the second equation:

$$14 = rf + 1.2\left(\frac{20 - rf}{1.8}\right)$$

Now, simplify and solve for rf :

$$\begin{aligned} 14 &= rf + \frac{24 - 1.2rf}{1.8} \\ &\Longrightarrow 14 - rf = \frac{24 - 1.2rf}{1.8} \\ &\Longrightarrow 1.8(14 - rf) = 24 - 1.2rf \\ &\Longrightarrow 25.2 - 1.8rf = 24 - 1.2rf \\ &\Longrightarrow 1.2 = 0.6rf \\ &\Longrightarrow rf = 2\%. \end{aligned}$$

Now, you can substitute $rf = 2\%$ back in any of stock A or B CAPM equations and solve for Er_M as follows:

$$\begin{aligned} 20 &= 2 + 1.8(Er_M - 2) \\ &\Longrightarrow 18 = 1.8Er_M - 3.6 \\ &\Longrightarrow 18 = 1.8Er_M - 3.6 \\ &\Longrightarrow 21.6 = 1.8Er_M \\ &\Longrightarrow Er_M = 12\%. \end{aligned}$$

You can double check now that if $rf = 2\%$ and $Er_M = 12\%$, then Er_A has to be 20% and Er_B has to be 14%.

A quick look at Figure 2, one can draw the following remarks on the SML and the CAPM in general: The first observation is related to the volatility of returns. If beta of an asset j is **equal to 1**, then from the CAPM equation, it must be that $Er_j = Er_M$. If, however, beta of an asset j is **equal to zero**, then $Er_j = rf$, which is also obvious from the CAPM equation. The interesting question that one might think of here is as follows: If beta of an asset is equal to zero, does this mean that the asset has no volatility? The answer is no. If beta of an asset is zero, it does not mean that the asset is completely free from any volatility (any risk). Zero beta implies that the systematic risk component is equal to zero. However, there exists a non-systematic risk component, also known as firm-specific or idiosyncratic risk, that the asset is still exposed to and, therefore, it is not completely free from volatility.[1] It is difficult to find such an asset in the market. Most of the stocks traded in the market have beta coefficients that are slightly above or below one. To give a confidence interval, we can say that most of the beta coefficients of the assets in the market range between 0.5 and 1.5. It is rare to find assets with zero or negative betas. The closest and, perhaps, the only example is the stocks of gold mining corporations. These stocks have betas closer to zero.

The second important **observation** on the SML and CAPM is that the SML is the line that describes the required rate of return on a **risky asset** as a function of the asset's risk measured by its **beta coefficient** and not by its **sigma** (standard deviation).

Finally, it is worth noting that CAPM is considered an equilibrium model of price formation. The financial equilibrium follows from the assumptions of CAPM. In particular, the assumption of perfect information (or market efficiency). This assumption implies that all investors will have the same expectations regarding assets' returns. If follows then that any expected return of any asset traded in the market must fall on the SML. If we further assume that lending is equal to borrowing, then the SML simply represents financial equilibrium in $Er - \beta$ space.

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[1] The word "idiosyncracy" is a Greek word. In general, it refers to an unusual trait of a person. It can also be used to express eccentricity or pecularity of a person. In linguistics, the term "idiosyncratic" refers to one thing for an individual. In the economic context, idiosyncratic shock of a security is a security-specific shock, i.e., a shock due to some specific factors relevant to the corporation or the financial entity that issued that particular security.

4. CAPM in Practice

In this section we focus on the application of CAPM to risky **portfolios** rather than just individual stocks. Eventually, we wish to show how CAPM can be used in practice to evaluate the performance of portfolio managers. But, first, let's see how the beta coefficient and the expected rate of return is calculated for a portfolio.

Consider a portfolio p that consists of n risky stocks with weights $w_1, w_2, ..., w_n$ such that the sum of the weights add to 1; that is,

$$w_1 + w_2 + \cdots + w_n = 1.$$

It can be shown that the beta of a **risky portfolio** of n assets is the weighted sum of the individual assets' betas; that is, for a portfolio that consists of n assets,

$$\beta_p = w_1\beta_1 + w_2\beta_2 + \cdots + w_n\beta_n, \tag{3.9}$$

with w_i being the weight of asset i in the portfolio and the sum of weights add to 1. This, in turn, implies that, for any arbitrary portfolio p, the CAPM holds; that is,

$$Er_p = rf + \beta_{p,M}(Er_M - rf). \tag{3.10}$$

The result in (3.9) implies that the CAPM can be used by portfolio managers to compute the required rate of return on **risky portfolios**. But, what about the conventional way of computing Er_p? Recall in Chapter 2, we discussed that the expected rate of return on a portfolio, Er_p, is calculated as a weighted sum of the expected rates of return on the assets forming it; that is,

$$Er_p = w_1Er_1 + w_2Er_2 + \cdots + w_nEr_n. \tag{3.11}$$

So, what is the difference between equation (3.10) and equation (3.11) ? Er_p in equation (3.10) is computed under the presumption that all CAPM axioms hold. Thus, equation (3.10) makes sense only if you assume that the axioms of CAPM hold. In practice, this is not always the case. Therefore, investors always compute the expected return on their portfolios using (3.11). Fund managers, however, use CAPM as a tool to assess their performance. That's why they opt to compute the expected rate of return on their portfolios using (3.10) and, then, compare that rate to the **average historical rate of return** on the fund. We will discuss this shortly. But, first, let's see how investors compute the beta coefficient and the expected return on their portfolios by examining the applicability of equations (3.9) and (3.11).

EXAMPLE 21. *Suppose that you are constructing a portfolio p that consists of two risky assets and a risk-free asset. Suppose that you allocated* 50% *of* *your wealth in the risk-free asset and 50% in the two assets below:*

***Asset** i*	Er (%)	β_i	w_i
$i = 1$	7.6	0.2	0.25
$i = 2$	12.4	0.8	0.25

Let the two assets be traded in a market M with $Er_M = 14\%$ and let the risk-free rate be $rf = 6\%$. What is the beta coefficient of your portfolio p and what is its expected rate of return Er_p? Here, we are computing β_p and Er_p for an investor (and not a fund manager). Thus, using equation (3.9), we have

$$\begin{aligned}\beta_p &= w_{rf}\beta_{rf} + w_1\beta_1 + w_2\beta_2 \\ &= 0.5 \times 0 + 0.25 \times 0.2 + 0.25 \times 0.8 \\ &= 0 + 0.05 + 0.2 \\ &= 0.25,\end{aligned}$$

where $\beta_{rf} = 0$ is the systematic risk of the risk-free asset, which is zero. Now, using (3.11), we can find the expected rate of return on the portfolio as the weighted sum of the expected returns on the risk-free asset, asset 1, and asset 2: that is,

$$\begin{aligned}Er_p &= w_{rf}rf + w_1Er_1 + w_2Er_2 \\ &= 0.5 \times 6\% + 0.25 \times 7.6\% + 0.25 \times 12.4\% \\ &= 8\%,\end{aligned}$$

or equivalently, we can use the CAPM equation:

$$\begin{aligned}Er_p &= rf + \beta_p(Er_M - rf) \\ &= 6 + 0.25(14 - 6) \\ &= 8\%.\end{aligned}$$

PROBLEM 9. *Suppose that you are constructing a portfolio p that consists of two risky assets and a risk-free asset. Suppose that you allocated 50% of your wealth in the risk-free asset and 50% in the two assets below:*

***Asset** i*	Er (%)	β_i	w_i
$i = 1$	7.6	0.2	0.25
$i = 2$	12.4	0.8	0.25

Let the market expected rate of return be $Er_M = 14\%$ and the risk-free rate be $rf = 6\%$. Now, suppose that you want an expected rate of return of 11%. Your strategy is to sell some of your holdings of the risk-free asset and use the proceeds to buy the market portfolio. If this is the way you decide to revise your portfolio, characterize this new portfolio; that is, find the weights of the assets in the portfolio and $(\beta_p)_{New}$, given that $(Er_p)_{New} = 11\%$.

SOLUTION 9. *Let x be the percentage of wealth subtracted from the risk-free asset and used to buy the market portfolio. Then,*

$$(Er_p)_{New} = (0.5 - x) \times rf + x \times Er_M + w_1 \times Er_1 + w_2 \times Er_2,$$

or equivalently,

$$11 = (0.5 - x)6 + 14x + 0.25(7.6) + 0.25(12.4).$$

Now, we isolate x in one side as follows:

$$\begin{aligned} 11 &= 3 - 6x + 14x + 5 \\ &\implies 3 = 8x \\ &\implies x = \frac{3}{8} = 0.375 \end{aligned}$$

Thus, the new strategy is such that 37.5% of wealth is invested in the market portfolio, 12.5% is invested in the risk-free asset, and 25% in Asset 1 and 25% in Asset 2. Finally, the beta of the new portfolio is calculated as follows:

$$\begin{aligned} \left(\beta_p\right)_{New} &= w_{rf}\beta_{rf} + w_M\beta_M + w_1\beta_1 + w_2\beta_2 \\ &= 0.125 \times 0 + 0.375 \times 1 + 0.25 \times 0.2 + 0.25 \times 0.8 \\ &= 0.625, \end{aligned}$$

where $\beta_M = 1$ always. Equivalently, we can also compute $\left(\beta_p\right)_{New}$ from CAPM equation; that is,

$$\begin{aligned} \left(Er_p\right)_{New} &= rf + \left(\beta_p\right)_{New}\left(Er_M - rf\right) \\ &\implies 11 = 6 + \left(\beta_p\right)_{New}(14 - 6) \\ &\implies \left(\beta_p\right)_{New} = \frac{5}{8} = 0.625. \end{aligned}$$

You can now check that $\left(Er_p\right)_{New}$ is indeed 11%. Using CAPM, we have:

$$\begin{aligned} \left(Er_p\right)_{New} &= rf + \left(\beta_p\right)_{New}\left(Er_M - rf\right) \\ &= 6 + 0.375(14 - 6) \\ &= 11\%. \end{aligned}$$

PROBLEM 10. *Suppose CAPM assumptions hold. You invested \$100,000 in the following stocks*

Stock	*Amount*	*Beta*
A	\$20,000	0.8
B	\$30,000	1.2
C	\$10,000	1.4
D	\$40,000	1.1

If the risk-free rate is 5 percent and the market risk premium is 8 percent, what is the expected return on your portfolio and what is its measure of systematic risk?

SOLUTION 10. *Let the constructed portfolio be p. Here we are trying to find Er_p and β_p. From CAPM applied to portfolios, we have*

$$\begin{aligned} Er_p &= rf + \beta_{p,M}\left(Er_M - rf\right) \\ &= 5 + 8 \times \beta_{p,M}. \end{aligned}$$

Now, we need to compute the beta coefficient of the portfolio. First, we find the weights of the four stocks in the portfolio. In particular, we have

$$\begin{aligned} w_A &= \frac{20,000}{100,000} = 0.2, \\ w_B &= \frac{30,000}{100,000} = 0.3, \\ w_C &= \frac{10,000}{100,000} = 0.1, \ \textit{and} \\ w_D &= \frac{40,000}{100,000} = 0.4. \end{aligned}$$

Then, we have

$$\begin{aligned} \beta_{p,M} &= w_A\beta_{A,M} + w_B\beta_{B,M} + w_C\beta_{C,M} + w_D\beta_{D,M} \\ &= 0.2(0.8) + 0.3(1.2) + 0.1(1.4) + 0.4(1.1) \\ &= 1.1 \end{aligned}$$

This is a measure of the portfolio's systematic risk. Thus,

$$Er_p = 5 + 8(1.1) = 13.8\%.$$

The result in equation (3.10) is an extremely important result as it extends the applicability of the CAPM from assessing the risk-return relationship of one single risky asset to a risky portfolio of n assets. It shows, as we will demonstrate shortly, that the SML can be used to assess the performance of portfolio managers. This is known as the **performance evaluation approach of CAPM**. The idea here is that if markets are working as they should be, any rational investor with basic market knowledge should be able to achieve an expected rate of return on the SML. Portfolio managers claim that they can do better in the sense that they can achieve a rate of return on their portfolio that is somewhere to the North West of the SML. Thus, any deviation from the SML in that direction can be used to assess the performance of portfolio managers. This deviation is known as **alpha** of the portfolio and it is measured by the equation

$$\begin{aligned} \alpha_p &= \overline{r}_p - RRR \text{ from CAPM} \\ &= \overline{r}_p - \underbrace{\left[rf + \beta_{p,M}\left(Er_M - rf\right)\right]}_{\text{Required rate of return from CAPM}}, \end{aligned} \tag{3.12}$$

where $\overline{r}_p$ is the **average rate of return** on the portfolio that is measured over the last few years and $\left[rf + \beta_{p,M}\left(Er_M - rf\right)\right]$ is the **required or expected rate of return** obtained from CAPM as in equation (3.10). CAPM predicts that the two rate are equal; that is, $\alpha_p = 0$ according to CAPM. This is due to the implicit assumption about market efficiency. In practice, however, markets are not efficient all the time. Therefore, well trained portfolio managers can construct portfolios that achieve rates of return that beat the market. If the average rate of return on a portfolio exceeds the required rate of return entailed by CAPM, the portfolio alpha will be positive, which signifies that the portfolio is achieving an excess return above the one predicted by the CAPM. This measure, i.e., the alpha of the portfolio, reflects the superiority of the fund's performance.

In the previous sections, we gave the intuition behind the CAPM and its graphical presentation (the SML). In this section, we will concentrate on the usefulness of CAPM as a financial equilibrium theory in practice. In particular, we will consider the use of the SML in measuring the performance of portfolio managers. As mentioned in the previous sections, positive deviations from the SML indicates good performance in managing a fund or a portfolio of risky assets. Any deviation from the SML in the North West direction can be used to assess the performance of portfolio managers. This deviation is the **alpha** of the portfolio, which is the difference between the **average rate of return** on the portfolio over the last few years and the **required rate of return** as suggested by CAPM (see equation (3.10)).

DEFINITION 30. *Suppose CAPM assumptions hold. The alpha of a portfolio fund, α_p, is the difference between the average rate of return on the portfolio and the required rate of return that is predicted by CAPM; that is,*

$$\alpha_p = \overline{r}_p - \underbrace{\left[rf + \beta_{p,M}\left(Er_M - rf\right)\right]}_{\text{Required rate of return}}.$$

CAPM predicts that $\alpha_p = 0$ due to market efficiency.

Perhaps it is best to illustrate this point by means of an example.

EXAMPLE 22. *Consider a fund with an average annual rate of return of 15.12% and a standard deviation of 3.2% over the period from January 2012 till December 2018. Suppose that the fund manager estimated the the beta of the fund during that period and found it to be 0.02. Historical data suggests that the risk-free rate of return is 5% and the market risk premium is 6%. We can use the CAPM to assess the performance of this fund manager as follows. First, note that the SML that corresponds to the previous data is*

$$Er_p = 5\% + \beta_p \times 6\%$$

as shown by the straight line originating from the risk-free rate on the vertical axis in Figure 5. Given that the beta of the portfolio is 0.02, the required rate of return on this fund is

$$Er_p = 5\% + (0.02 \times 6\%) = 5.12\%.$$

This is the rate of return that the manager should have achieved on his or her fund if CAPM assumptions hold, i.e., if markets are efficient. But, what the manager actually did earn over the last few years is $\overline{r}_p = 15.12\%$. This figure represents the average rate of return on the fund, which is a measure of its actual performance. This leaves 10% extra return due to the talent of *the fund manager (see the dashed line in Figure 5). In other words, this is the value added that any client would receive if he or she decided to use the service of this fund manager. This value added is known as "alpha" of the portfolio and is computed as*

$$\alpha_p = \overline{r}_p - Er_p = 15.12\% - 5.12\% = 10\%.$$

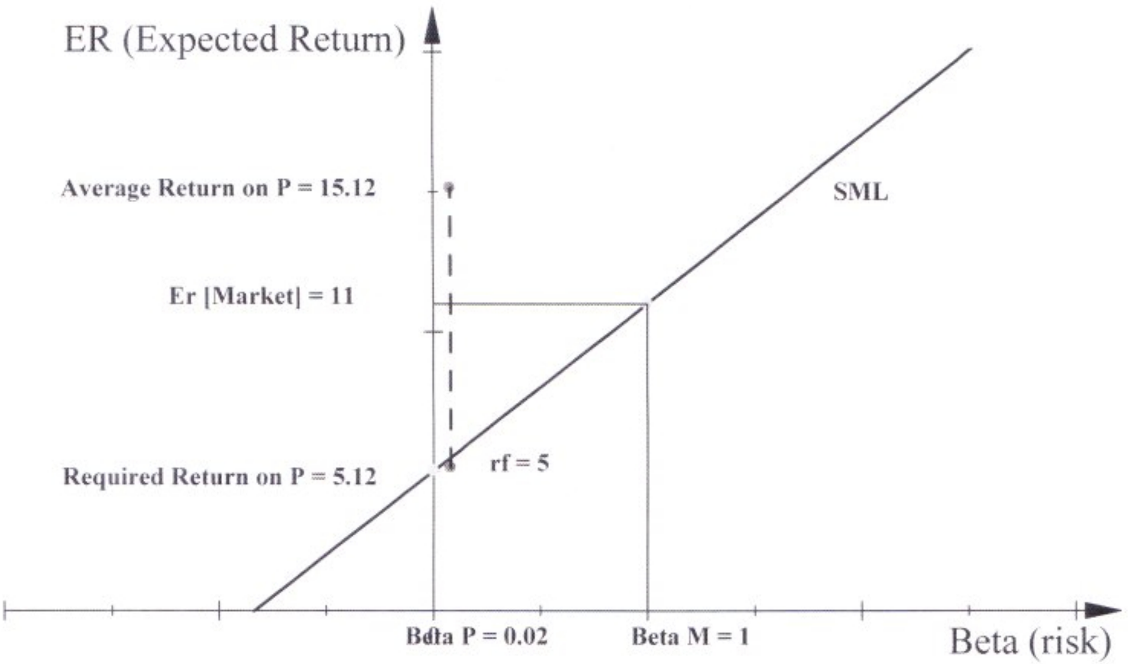

Figure 5: Performance evaluation approach of CAPM: Alpha of the portfolio.

Obviously, according to the previous analysis, the portfolio manager can claim an excellent performance in managing the fund. It should be noted, however, that the positive alpha of 10% reflects other types of risks that are not explained by the CAPM. Keep in mind that the only risk that the CAPM can capture is the systematic risk (or the market risk). There exists, however, other types of risk that are not captured by the beta coefficient of the CAPM. Thus, high excess return on a portfolio comes with higher risk that the fund manager has to bear.

5. Practice Problems

(1) **Calculating Beta**. Following are several months' rates of return for Tumblehome Canoe Company, r_T, and the market rate of return, r_M. Compute Tumblehome beta (Adapted from Brealey et

al (2009))

Month	r_M (%)	r_T (%)
1	0	+1
2	0	−1
3	−1	−2.5
4	−1	−0.5
5	+1	+2
6	+1	+1
7	+2	+4
8	+2	+2
9	−2	−2
10	−2	−4

(2) An economy has two scenarios: boom or bust. The returns in each scenario for the market portfolio, an aggressive stock A, and a defensive stock D are

		Rate of return	**Rate of return**
Scenario	**Market**	**Aggressive Stock A**	**Defensive Stock D**
Bust	−8%	−10%	−6%
Boom	32%	38%	24%

- (a) If each scenario is equally likely, find the beta of each stock.
- (b) Find the expected rate of return on the market portfolio and on each stock.
- (c) If the treasury bill rate is 4%, what does CAPM say about the fair expected rate of return on the two stocks?

(3) An investor borrowed $1000 at the risk-free rate of 5%. The investor invested the borrowed money and his own money of $4000 in a market portfolio with an expected rate of return of 15%. What is the expected rate of return on the portfolio that was constructed by the investor?

(4) Answer the following questions
- (a) State the axioms of CAPM.
- (b) Define what it means by the Efficient Markets Hypothesis.
- (c) If the expected rate of return on the market portfolio is 14% and the risk-free rate is 6%, find the beta for a portfolio p that has expected rate of return of 10%.
- (d) Illustrate your answer by sketching the SML showing the portfolio that you constructed on the line.
- (e) What percentage of this portfolio must you put into the market portfolio and what percentage must you put in the risk-free asset in order to achieve an expected return of 10%? (Hint: notice that $w_{rf} + w_M = 1$. So, you can use $w_M = 1 - w_{rf}$ and solve for w_{rf})

(5) Suppose that the TSX, with a beta of 1, has an expected return of 10 percent and Treasury bills provide a risk-free return of 5 percent.

   (a) *Construct a portfolio from these two assets with an expected return of 8 percent. What is the beta of this portfolio?*
   (b) Construct a portfolio from these two assets with a beta of 0.4. Calculate the portfolio's expected return.

(6) **Systematic and non-systematic risk**. Consider the standard CAPM framework and assume that all the assumptions regarding CAPM hold. Suppose that historical data shows that the risk-free rate is 5%, the expected rate of return on the market is 15%, and the standard deviation of the market rate of return is 0.2. Answer the following questions:
   (a) What is the beta coefficient of a portfolio p with $Er_p = 20\%$.
   (b) Find a measure of the systematic risk of this portfolio.

(7) **CAPM assumptions and implication**. If the expected rate of return on the market portfolio is 14% and the risk-free rate is 6%, answer the following questions:
   (a) Find the beta for a portfolio that has expected rate of return of 10%. What **assumptions** concerning this portfolio and/or market conditions do you need to make to calculate the portfolios' beta? What is the main **implication** of this set of assumptions?
   (b) What percentage of this portfolio must you put into the market portfolio in order to achieve an expected return of 10%? Illustrate your answer by sketching the SML showing the portfolio that you constructed on the line.

(8) Suppose you are given the following information regarding two stocks: A and B as follows:

$$Er_A = 10.5\% \qquad \beta_A = 0.7$$

$$Er_B = 13\% \qquad \beta_B = 1.2$$

   (a) What is the beta coefficient of a stock D with an expected rate of return of 12%? From your finding, what can you say about stock D?
   (b) Suppose you decided to invest 30% of your wealth in A, 30% in B, and 40% in the risk-free rate. What is the expected rate of return and the beta of the constructed portfolio?
   (c) Suppose you decided to invest 125% of your wealth in A and the remainder in the risk-free rate via borrowing at the risk-free rate. What is the expected rate of return and the beta of the constructed portfolio?
   (d) A portfolio fund manager, based on his fund's past performance, claims that the average rate of return on the fund is 15% and the fund beta is 0.875. How can we use the CAPM to assess the performance of this fund manager?

(9) **CAPM & SML**. Consider the CAPM framework. Suppose that you currently have 50% of your wealth in a risk-free asset and 50%

in the four assets below:

Asset i	Er (%)	β_i	w_i
$i = 1$	7.6	0.2	0.1
$i = 2$	12.4	0.8	0.1
$i = 3$	15.6	1.2	0.1
$i = 4$	18.8	1.6	0.2

Let the four assets be traded in a market M with $Er_M = 14\%$ and let the risk-free rate be $rf = 6\%$. Answer the following questions.

(a) Identify the current (old) portfolio; that is, find the weights, the expected rate of return and the beta of that portfolio. Since the weights are given, you need to find $(Er_p)_{\text{Old}}$ and $(\beta_p)_{\text{Old}}$.

(b) Now, suppose that you want an expected rate of return of 12%. Your strategy is to sell some of your holdings of the risk-free asset and use the proceeds to buy the market portfolio. If this is the way you decide to revise your portfolio, characterize this new portfolio; that is, find the weights, $(Er_p)_{\text{New}} = 12\%$, and $(\beta_p)_{\text{New}}$.

(c) Now, suppose, instead, that you want to hold only the risk-free asset and the market portfolio. What set of weight would give you an expected 12% rate of return? What is the beta of this new portfolio?

(10) **On the definition of the beta coefficient**. Consider again the CAPM framework. Assume that investors can borrow/lend at the risk-free rate. Suppose an investor forecasted the correlation coefficient between the rate of return on WAT Mutual Fund, W, and the market portfolio, M, at 0.4. The investor's forecast of the standard deviations of the rates of returns are 0.25 and 0.20 for W and M, respectively. Suppose that the investor wishes to combine the WAT Fund with a risk-free security to obtain a portfolio, p, with a beta of 1.6. Find the weights of W and the risk-free asset in this portfolio. Explain whether the investor needs to borrow or lend in that case and how he or she would do this; that is, explain the investor's strategy to construct such a portfolio.

CHAPTER 4

Time Value of Money

In this chapter, we wish to explore the value of money over time. Any investment decision involves paying an amount on the spot to acquire an asset in hope of achieving financial gain at a future time. Companies, for instance, invest in machines and equipment (tangible assets) in hopes of receiving more money from the use of these assets. Investment could also be made in other intangible assets such as patents or trademarks. Again, the whole idea is to invest an amount today in hopes of achieving some return or gain in the future. Individuals invest in financial assets as well. The decision to invest your money in financial instruments, e.g., bonds and stocks, in hope of achieving some return in the future is a key in point. In short, whether it is a company or an investor, the entire financial plan is based on **time**; when the time element is not present, the future doesn't exist, and consequently, investors have no motive to buy assets in the first place. The objective of this chapter is to familiarize the reader with the role played by time in financial transactions or financial contracts. In particular, we discuss how to calculate the future value to which current money invest will grow, the present value of a future payment, and to how compute the interest rate implied by the present value or the future value.

1. Future Values

Consider this simple financial contract. Suppose you seek to invest \$100 in a bank account. The bank is currently paying an interest rate of 10% per year on deposits. We wish to compute the value of this \$100, that was deposited *today, at the end of the year. Let FV denote the future value, PV denote* the present value, r be the interest rate, and I be the interest payment. In this example, $PV = 100$, $r = 0.1$, and FV is what we wish to find. The interest payment is simply the interest rate times the amount invested; that is,

$$\begin{aligned} I &= r \times PV \\ &= 0.1 \times 100 = \$10. \end{aligned} \tag{4.1}$$

Thus, by the end of the year, you should have \$100 + \$10 in your bank account; that is, the future value is simply the present value plus the interest payment:

$$FV = PV + I. \tag{4.2}$$

Substituting the definition of I in equation (4.1) back in (4.2) yields the future value formula for an amount growing at a rate r for one year:

$$FV_{\text{at the end of year 1}} = PV\,(1+r)\,.$$

Suppose you are at the end of year 1. If you made the decision to leave your money in the bank for a second year, the current balance, which is \$110 now, will grow for another year at a rate $r = 10\%$. The future value in this case will be

$$\begin{aligned} FV_{\text{at the end of year 2}} &= 110 + (0.1 \times 110) \\ &= 110(1+0.1) \\ &= 100(1+0.1)(1.01) \\ &= 100(1+0.1)^2 = 121 \end{aligned}$$

Thus,

$$FV_{\text{at the end of year 2}} = PV\,(1+r)^2\,.$$

In the same way, we can easily see that the future value of a PV after n years growing at an annual rate r is calculated by the following formula

$$FV = PV(1+r)^n. \tag{4.3}$$

Calculating the future value from the future value equation in (4.3) is easy using almost any calculator. If you don't have a calculator, you can use a table of future values. This table shows the future value of a \$1 investment for various interest rates and investment periods. The future value of a \$1 investment at r percent per period for n periods is known as the **future value interest factor (FVIF)** and is usually denoted as $FVIF(r,n)$. Thus,

$$FVIF(r,n) = (1+r)^n\,. \tag{4.4}$$

Future value tables are tedious and they usually show future values for only a limited set of interest rates and time periods. The power key on your calculator is faster and easier. Alternatively, you can use financial calculator or spreadsheets.

Notice that, in the previous example, your interest income in the first year is

$$I_{\text{in year 1}} = 0.1 \times 100 = \$10$$

and in the second year

$$I_{\text{in year 2}} = 0.1 \times 110 = \$11.$$

Your income at the end of the second year is \$121 (\$100+\$10+\$11) is higher because your are earning interest on interest. This is called **compounding** or **compound interest**. If, on the other hand, the bank calculated your interest based on your original investment, you would be paid simple interest. With **simple interest**, your balance at the end of the year would be \$120 ($\$100 + 2 \times \$10$).

EXAMPLE 23. *Consider the following investment opportunity: Lend a borrower today \$100 and receive, after two years, \$120. What will be your rate of return on investment if you accept this deal? In this example, we have an amount today* $PV = 100$ *and a future amount* $FV = 120$ *in* $n = 2$ *years. The interest rate on this investment opportunity,* r*, is the rate of return that you will make as an investor. Notice how* r *here has a different meaning; it is the rate of return on your money and not the bank rate. Now, we wish to find that rate. Using the* FV *formula, we know that*

$$FV = PV(1+r)^n,$$

where $PV = 100$, $FV = 120$, $n = 2$, *and* r *is the only unknown in the formula. Now, we solve for* r *as follows:*

$$\begin{aligned} FV &= PV(1+r)^n \\ &\Longrightarrow 120 = 100(1+r)^2 \\ &\Longrightarrow \frac{120}{100} = (1+r)^2 \\ &\Longrightarrow 1.2 = (1+r)^2 \\ &\Longrightarrow (1.2)^{\frac{1}{2}} = 1+r \\ &\Longrightarrow (1.2)^{\frac{1}{2}} - 1 = r \\ &\Longrightarrow r = (1.2)^{\frac{1}{2}} - 1 = 0.095(\mathit{or}\ 9.5\%). \end{aligned}$$

Thus, your rate of return if you choose this investment opportunity will be 9.5%.

Before we turn our attention to present values, it is worth noting that, in the context of computing the FV of a PV, the interest rate r in equation (4.3) is thought of as a **growth rate**. It is the rate by which the current amount grows over time. As we will discuss in the next subsection, r has a different meaning when computing the PV of a FV; it is the **discount rate**.

2. Present Values

Future value is calculated by multiplying the present value by one plus the interest rate. Then, to compute the present value, we simply divide the future value by one plus the interest rate. This is easy to see from the simple manipulation of equation (4.3):

$$PV = \frac{FV_{\text{after } n \text{ periods}}}{(1+r)^n}. \tag{4.5}$$

The present value is the value today of a future cash flow. The interest rate r in the present value formula in equation (4.5) is known as the **discount rate** and the present value is sometimes called the **discounted value** of the future payment.

EXAMPLE 24. *Suppose you are planning to buy a new computer worth \$4000 after 1 year. If the interest rate on deposits at the bank is 6%, how much money should you set aside today in order to pay for your purchase in one year? Here* $FV = \$4000$, $n = 1$, $r = 0.06$. *Then, the present value is easily obtained by direct substitution in equation (4.5) as follows:*

$$PV = \frac{4000}{(1+0.06)^1} = \$3773.6.$$

Notice that investing \$3773.6 *at* 6% *yields* \$4000 *in one year. Thus,* \$3773.6 *is the discounted value of a \$4000 future payment.*

It is often useful to express the present value formula in equation (4.5) as follows

$$PV = FV \times \frac{1}{(1+r)^n}. \tag{4.6}$$

The term $1/(1+r)^n$ is called the **discount factor** or the **preset value interest factor (PVIF)**. It measures the present value of \$1 to be received in n years from today at a discount rate r percent.

$$PVIF(r,n) = \frac{1}{(1+r)^n}. \tag{4.7}$$

The simplest way to find the PVIF is to use the calculator or spreadsheets. Some financial managers, however, find it convenient to use tables of discount factors.

EXAMPLE 25. [1] *Kangaroo Autos is offering free credit on a \$10,000 car. You pay \$4,000 down and then the balance at the end of 2 years. Shark Motors next door does not offer free credit but will give you \$500 off the list price. If the interest rate is 10 percent, which company is offering a better deal? In this example, we have two different pattern of cash payment. In such a case, we should compute the present value of payments in each alternative. Using the time line below, we can find the present value of the payments to Kangaroo by adding the \$4,000 paid on the spot to the present value of the*

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[1] Adapted from Brealey et al (2006).

\$6,000 owed in 2 years.

$$\begin{array}{ccccc} 4000 & & & & 6000 \\ \uparrow & & & & \uparrow \\ 0 & ---- & 1 & ---- & 2 \end{array}$$

$$\begin{array}{l} a = 4000 \\ b = \frac{6000}{(1+0.1)^2} \quad \longleftarrow \quad \longleftarrow \quad \longleftarrow \quad \hookleftarrow \\ PV = a + b \end{array}$$

Notice that the \$6,000 payment is due at the end of year 2. To come up with 6,000 in two years at a rate of 10%, you will need $6000/(1+0.1)^2$ *today. Thus, the PV of payments to Kangaroo is*

$$PV_{Kangaroo} = 4000 + \frac{6000}{(1+0.1)^2} = \$8958.68$$

The present value of payments to Shark is

$$PV_{Shark} = 10000 - 500 = \$9500.$$

Thus, Kangaroo is offering a better deal. The point of this example is to never compare cash flows occurring at different times without discounting them to a common date.

The PV formula in (4.3) or the FV formula in (4.5) can be used to find the interest rate or the investment period. The following examples illustrate these cases.

EXAMPLE 26. *Suppose your friend proposed to borrow from you a \$1,000 today and promised to pay you back \$1,200 in two years. If the interest rate on bank deposits is 9%, would you rather invest your money in the bank or accept your friend's proposal? In this example, the opportunity cost of your money is 9%. You need to calculate the rate of return, i.e., the interest rate, that you would achieve if you decided to accept your friend's proposal. Using the PV formula, we have*

$$1000 = \frac{1200}{(1+r)^2}.$$

This is one equation in one unknown, r. You simply need to solve for the interest rate as follows:

$$\begin{aligned} (1+r)^2 \quad &= \quad \frac{1200}{1000} \implies (1+r) = \left(\frac{1200}{1000}\right)^{\frac{1}{2}} \\ &\implies \quad r = \left(\frac{1200}{1000}\right)^{\frac{1}{2}} - 1 = 0.095. \end{aligned}$$

Thus, you should accept your friend's deal since the rate of return on this investment is 9.5%, which is higher than the rate that you can otherwise achieve by depositing your money in the bank.

EXAMPLE 27. *Consider the previous example. Assume that everything is known except for n; that is,*

$$1000 = \frac{1200}{(1+0.095)^n}.$$

We can easily find n as follows:

$$(1+0.095)^n = \frac{1200}{1000}$$

Take the natural logarithm on both sides and remember the rule: $\log x^a = a \log x$. This gives

$$\begin{aligned} n\log(1+0.095) &= \log\left(\frac{1200}{1000}\right) \\ &\Longrightarrow \quad n = \frac{\log\left(\frac{1200}{1000}\right)}{\log(1+0.095)} = 2. \end{aligned}$$

3. PV and FV of Multiple Cash Flows

In the previous sections, we looked at problems involving one single cash flow. In this section we consider investments characterized by a stream of cash flows. In this type of problems, drawing a time line is very useful in computing the present value or the future value of the cash flows.

EXAMPLE 28. *Suppose you are planning to buy a computer in two years. You decided to set aside, or save, \$1,000 today, \$1,200 at the of year 1, and \$1,400 at the end of year 2 at an interest rate of 10%. How much will you be able to spend on a computer in three years? In this example, we basically want to find the future value of a stream of cash flows. Using the future value formula, we can simply bring all cash flows to the beginning of year 3 as follows:*

$$\begin{array}{ccccccc}
1000 & & 1200 & & 1400 & & FV = ? \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & ---- & 1 & ---- & 2 & ---- & 3 \\
 & & & & \hookrightarrow & \longrightarrow & a = 1400(1.1) \\
 & & \hookrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & b = 1200(1.1)^2 \\
\hookrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & c = 1000(1.1)^3 \\
 & & & & & & FV = a + b + c
\end{array}$$

$$FV = 1000(1+0.1)^3 + 1200(1+0.1)^2 + 1400(1+0.1)$$

EXAMPLE 29. [2] *Suppose that your auto dealer gives you a choice between paying \$15,500 for a new car or entering into an installment plan where you pay \$8,000 down today and make payments of \$4,000 in each of the*

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[2] Adapted from Brealey et al (2006)

next 2 years. Which is the better deal assuming that the interest rate you can earn on a safe investment is 8%? If you compare 15,500 and 16,000 as the total payments under the two plans, you are completely ignoring the time value of money. The right way of comparing two plans that involve different payments at different times is to discount the payments and compare the present values. The present value of the first plan is simply \$15,500 today. Now, let's find the present value of the second plan. Again, we can use the time line to discount all future payments pertaining to this plan using 8% interest rate as follows:

$$\begin{array}{ccccc} 8000 & & 4000 & & 4000 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & ---- & 1 & ---- & 2 \end{array}$$

$$\begin{array}{rcccc} a = 8000 & & & & \\ b = \frac{4000}{(1+0.08)^1} & \longleftarrow & \hookleftarrow & & \\ c = \frac{4000}{(1+0.08)^2} & \longleftarrow & \longleftarrow & \longleftarrow & \hookleftarrow \\ PV = a + b + c & & & & \end{array}$$

$$PV = 8000 + \frac{4000}{(1+0.08)^1} + \frac{4000}{(1+0.08)^2} = 15,133.06$$

Thus, you are better off taking the second plan.

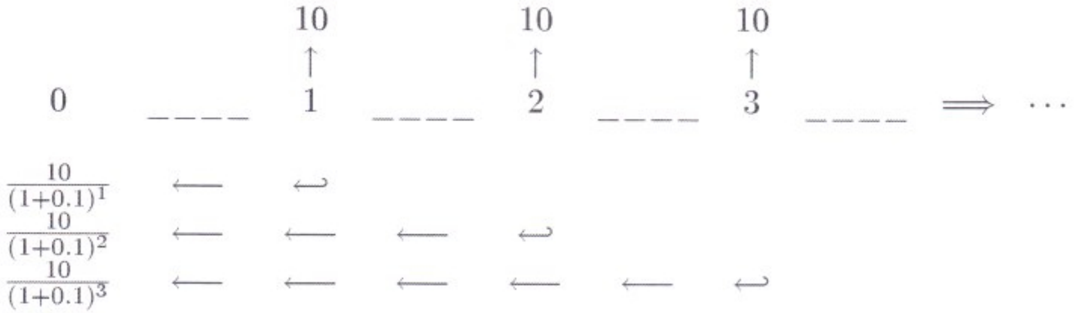
4. Perpetuities and Annuities

An annuity is a stream of cash flows that is equally spaced. A perpetuity is a stream of cash flows that never ends. For instance, a 25-year mortgage that require equal monthly payments is an annuity. A preferred share that promises the holder fixed dividends for ever is a perpetuity. Our task in this section is to evaluate these types of cash flows.

4.1. PV of Perpetuities. Sometimes governments borrow by issuing perpetuities. The government pays the investors holding these perpetuities a fixed payment for ever. Let $F = \$100$ be the **face value** of the government bond. This is the amount that the investor must pay, which is also the amount that the government borrows, in order to own the bond. Suppose that the bond promises an interest rate $r = 10\%$ every period for ever. The interest payment or cash payment, denoted by I, is \$10 ($10\% \times 100$); that is, the cash payment is calculated from the following formula:

$$I = r \times F.$$

This perpetuity then guarantees the investor \$10 each period for ever. What is the present value of this perpetuity?



A quick look at the time line above, one can see that the PV is simply the discounted value of the fixed payment of \$10 for ever; that is,

$$PV = \frac{10}{(1+0.1)^1} + \frac{10}{(1+0.1)^2} + \frac{10}{(1+0.1)^3} + \cdots$$

How can we evaluate this progressive sum? Luckily, there is a neat and simple mathematical trick that we can use to derive a simple formula to value any perpetuity. Here is how it is done:

$$PV = \frac{I}{(1+r)^1} + \frac{I}{(1+r)^2} + \frac{I}{(1+r)^3} + \cdots \tag{4.8}$$

Take I common factor and re-write equation (4.8) as follows:

$$PV = I\left[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots\right]$$

Notice that 1 raised to any power is still 1. Thus,

$$PV = I\left[\underbrace{\gamma + \gamma^2 + \gamma^3 + \cdots}_{S}\right], \tag{4.9}$$

where $\gamma = \frac{1}{1+r}$ is the discount factor. Now, let S denote the sum on the right-hand-side of (4.9). If we deduct γS from S, we simply obtain γ since all the terms with exponents higher than one cancel out; that is,

$$\begin{array}{cccc} & S = & & \gamma + \gamma^2 + \gamma^3 + \cdots \\ - & \gamma S = & & \gamma^2 + \gamma^3 + \cdots \\ & \text{-----------} & & \text{--------------} \\ \Longrightarrow & S - \gamma S = & & \gamma \end{array}$$

Thus,

$$S(1-\gamma) = \gamma,$$

or equivalently,

$$S = \frac{\gamma}{1-\gamma} = \frac{\frac{1}{1+r}}{1-\frac{1}{1+r}} = \frac{\frac{1}{1+r}}{\frac{r}{1+r}} = \frac{1}{r}.$$

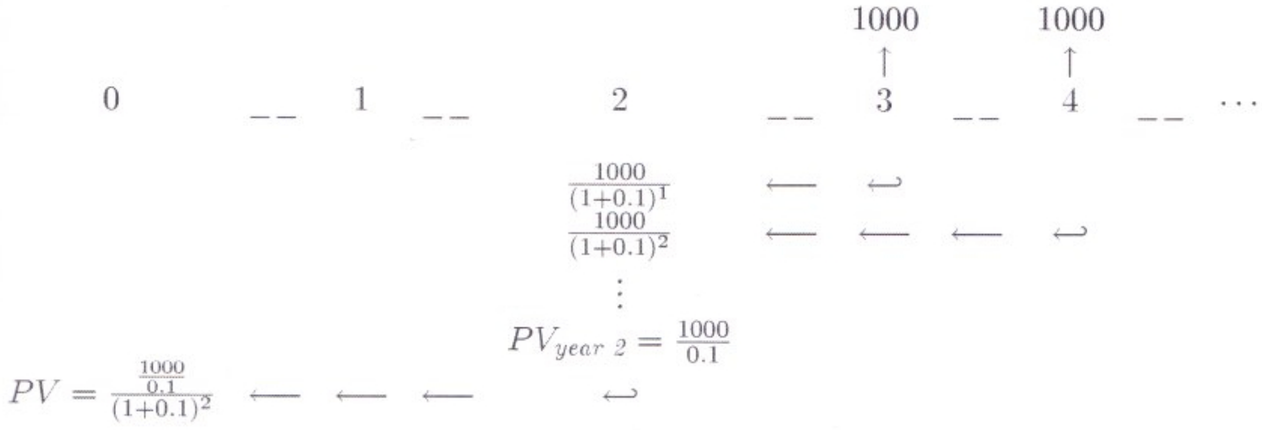
Substituting $\frac{1}{r}$ back in equation (4.9), we obtain the very simple formula

$$PV = \frac{I}{r}. \tag{4.10}$$

Thus, the present value of any perpetuity is simply the fixed cash payment divided by the interest rate (or the discount rate).

Sometimes a perpetuity doesn't start to make payments for several years. You can still invoke the general present value formula for a perpetuity in (4.10) at the starting year of the perpetuity. But, then you ought to bring the PV back to year zero using the conventional discounting formula. The following example illustrates this case.

EXAMPLE 30. *Consider a perpetuity that pays out \$1000 a year with the first payment 3 years from now. If the discount rate $r = 0.1$, what is the present value of this perpetuity? Again, the time line is extremely useful in this type of problems.*


1000
1000
0
1
2
3
4
...
$\frac{1000}{(1+0.1)^1}$
$\frac{1000}{(1+0.1)^2}$
$\vdots$
$PV_{year\ 2} = \frac{1000}{0.1}$
$PV = \frac{\frac{1000}{0.1}}{(1+0.1)^2}$

A quick look at the time line above, one can notice that the present value formula of a perpetuity gives the discounted value of all infinite payments at the beginning of year 2; that is,

$$PV_{year\ 2} = \frac{1000}{0.1} = 10,000.$$

To bring this amount back to year 0, we ought to discount it for two years. Thus, the PV of this plan is

$$PV = \frac{PV_{year\ 2}}{(1+r)^2} = \frac{\frac{1000}{0.1}}{(1+0.1)^2}.$$

4.2. PV of Annuities. Now, let's turn our attention to annuities. Consider an annuity that pays a fixed cash flow, CF, every year/period for n years. Let r be the discount rate. How can we compute the PV of such an annuity? We will perform the same mathematical trick as in perpetuities. The only difference is that n in our case doesn't go for ever.

That being said, we can write the PV of all future discounted cash flows as follows:

$$\begin{aligned} PV &= \left[\frac{CF}{(1+r)^1} + \frac{CF}{(1+r)^2} + \frac{CF}{(1+r)^3} + \cdots + \frac{CF}{(1+r)^n}\right] \\ &= CF\left[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^n}\right] \\ &= CF\left[\underbrace{\left(\frac{1}{1+r}\right) + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^3 + \cdots + \left(\frac{1}{1+r}\right)^n}_{S}\right] \end{aligned}$$

Let $\gamma = \frac{1}{1+r}$ be the discount factor, then the sum of the progressive term on the right-hand-side is

$$S = \gamma + \gamma^2 + \gamma^3 + \cdots + \gamma^n$$

and

$$\gamma S = \gamma^2 + \gamma^3 \cdots + \gamma^{n+1}$$

Thus,

$$\gamma S - S = \gamma^{n+1} - \gamma,$$

or equivalently,

$$S(\gamma - 1) = \gamma(\gamma^n - 1) \implies S = \frac{\gamma(\gamma^n - 1)}{\gamma - 1}.$$

Substituting the definition of γ in the sum S and simplifying yields

$$S = \frac{\frac{1}{1+r}\left(\frac{1}{(1+r)^n} - 1\right)}{\frac{1}{1+r} - 1} = \frac{\frac{1}{1+r}\left(\frac{1}{(1+r)^n} - 1\right)}{\frac{1-1-r}{1+r}} = \frac{1}{r} - \frac{1}{r(1+r)^n}.$$

Thus, the PV of an annuity that pays CF every period for n periods is

$$PV = CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] \tag{4.11}$$

What about the future value of an annuity? Well, there is no need to derive the future value formula from scratch. Simply recall that

$$FV = PV(1+r)^n$$

and substitute the PV formula (equation (4.11)) back in the FV formula to obtain

$$FV = CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right](1+r)^n.$$

This is the future value of an annuity with cash flow CF, interest rate r, over n periods. Here is an example.

EXAMPLE 31. *A couple thinking about retirement decide to put aside \$3,000 each year in a savings plan that earns 8 percent interest. How much money will they have accumulated 30 years from now? We have an annuity with* $CF = 3000$ *for* $n = 30$ *years at an interest rate* $r = 8\%$. *The* FV *formula is simply the conventional* PV *formula of any annuity times* $(1+r)^n$*; that is,*

$$\begin{aligned} FV &= CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right](1+r)^n \\ &= 3000 \times \left[\frac{1}{0.08} - \frac{1}{0.08(1+0.08)^{30}}\right](1+0.08)^{30} \\ &= \$339,850. \end{aligned}$$

REMARK 7. *The perpetuity and annuity formulas assume that the first payment occurs at the end of the period. Sometimes, however, streams of cash payments start immediately. In such a case, the annuity is called an* ***annuity due****. The computation is essentially the same; the only difference is that the first payment is made immediately. The distinction between an annuity due and an ordinary annuity is best made by consulting the time line. Here is an example of a three-year ordinary annuity versus a three-year annuity due:*

$$\begin{array}{lcccccccc}
\textit{Ordinary Annuity} & & & CF & & CF & & CF \\
 & & & \uparrow & & \uparrow & & \uparrow \\
 & 0 & ---- & 1 & ---- & 2 & ---- & 3 \\
 & a = \frac{CF}{(1+r)^1} & \longleftarrow & \hookleftarrow & & & & \\
 & b = \frac{CF}{(1+r)^2} & \longleftarrow & \longleftarrow & \longleftarrow & \hookleftarrow & & \\
 & c = \frac{CF}{(1+r)^3} & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \hookleftarrow \\
PV = a + b + c & & & & & & &
\end{array}$$

$$\begin{array}{lccccccc}
\textit{Annuity Due} & CF & & CF & & CF & & \\
 & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 & ---- & 1 & ---- & 2 & ---- & 3 \\
 & \downarrow & & & & & & \\
 & a = CF & & & & & & \\
 & b = \frac{CF}{(1+r)1} & \longleftarrow & \hookleftarrow & & & & \\
 & c = \frac{CF}{(1+r)^2} & \longleftarrow & \longleftarrow & \longleftarrow & \hookleftarrow & & \\
PV = a + b + c & & & & & & &
\end{array}$$

Notice that the annuity due in this case is simply a 2-year ordinary annuity plus an immediate payment. Thus, the present value of an ordinary annuity is calculated from the following formula:

$$PV|_{\textit{annuity due}} = CF_0 + CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^{n-1}}\right],$$

where CF_0 is the cash flow at time $t = 0$. Notice that the present value of an annuity due over n years is simply the sum of today's cash flow, CF_0, and the present value of an ordinary annuity for $n - 1$ years.

PROBLEM 11. *Your landscaping company can lease a truck for \$8200 a year (paid at year-end) for 5 years. It can buy the truck for \$40,000. The truck will be valueless after five years. (a) If the interest rate your company can earn on its funds is 7 percent, is it cheaper to buy or lease? (b) If the lease payments are an annuity due, is it cheaper to buy or lease?*

SOLUTION 11. *First, consider the lease option as an ordinary annuity. Here, we have $n = 6$, $r = 7\%$, $CF = 8200$ a year, and we are trying to compute the PV of this ordinary annuity.*

$$\begin{array}{ccccccc}
 & & CF = 8200 & & CF = 8200 & & CF = 8200 \\
 & & \uparrow & & \uparrow & \cdots & \uparrow \\
0 & ---- & 1 & ---- & 2 & ---- & 6 \\
 & & & & & & \\
a & \longleftarrow & \hookleftarrow & & & & \\
b & \longleftarrow & \longleftarrow & \longleftarrow & \hookleftarrow & \cdots & \\
\vdots & \vdots & & & & \vdots & \\
f & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \hookleftarrow
\end{array}$$

$$PV = a + b + \cdots + f,$$

or equivalently, using the PV formula for an ordinary annuity, we have

$$\begin{aligned}
PV &= CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] \\
&= 8200 \times \left[\frac{1}{0.07} - \frac{1}{0.07(1+0.07)^6}\right] \\
&= \$39085.
\end{aligned}$$

The buy option will cost \$40,000. Therefore, the least option is cheaper.

(b) Now, let's consider the case where the lease is an ***annuity due****. The time line, in this case, should be:*

$$\begin{array}{cccccccc}
\textit{Annuity Due} & CF & & CF & & CF & & \\
 & \uparrow & & \uparrow & \cdots & \uparrow & & \\
 & 0 & ---- & 1 & ---- & 5 & ---- & 6
\end{array}$$

This means, we have a cash flow at time $t = 0$, denoted CF_0, amounting \$8200, and an ordinary annuity for $n - 1 = 5$ years. Thus, the present value *of an annuity due is*

$$\begin{aligned} PV &= CF_0 + CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^{n-1}}\right] \\ &= 8200 + 8200 \times \left[\frac{1}{0.07} - \frac{1}{0.07(1+0.07)^5}\right] \\ &= \$8200 + \$33622 \\ &= \$41822 \end{aligned}$$

Therefore, the lease option is more expensive than buying the truck. It follows, then, that in this case, one should consider buying the truck rather than leasing.

5. Effective Annual Interest Rates

So far, we were working under the presumption that payments are made annually and the interest is compounded annually. In other words, we have used **annual** interest rates to value a series of **annual** cash flows. In practice, however, interest rates may be quoted for days, months, or years. In the same way that compounding could occur more frequently than annually, payments could also be made at intervals other than annually. The rule of thumb is to ensure consistency between the interest rate and the number of periods. For instance, if interest rate is quoted per month rather than per year, then you must define the number of periods in your future value calculations as number of months. When the interest rate is quoted in a frequency other than annually, we ought to find the equivalent annual rate. This rate is known as the **effective annual interest rate** (**EAR**).

DEFINITION 31. *The **effective annual interest rate** (**EAR**) is the interest rate that is annualized using compound interest rate.*

For example, suppose you can borrow \$100 from a credit card company at a rate of 2 percent per month. Notice here that the quoted rate is 2% monthly. If you borrow this amount for one year, i.e., $n = 12$ months, then you will need to repay $100 \times (1+0.02)^{12} = \$126.8$. Thus, over one year, your debt grew from \$100 to \$126.8; that is, the EAR, or the **annually compounded rate**, is 26.8 percent. Notice that the effective annual equivalent to a monthly rate of interest is

$$1 + EAR = (1 + \text{ monthly rate})^{12} \Longleftrightarrow 1 + 0.268 = (1 + 0.02)^{12} \tag{4.12}$$

What about multiplying 2% a month times 12 months? This way of annualizing interest rate uses **simple interest** rather compound interest. The resulting 24% rate is known as the **annual percentage rate** (**APR**).

DEFINITION 32. *The* ***annual percentage rate (APR)*** *is the interest rate that is annualized using simple interest.*

Since the monthly rate is $\frac{24\%}{12} = 2\%$, i.e.,

$$\frac{APR}{m} = \frac{24\%}{12} = \text{monthly rate},$$

then equation (4.11) could be written as

$$1 + EAR = \left(1 + \frac{APR}{12}\right)^{12},$$

or more generally, for any compounding period m,

$$1 + EAR = \left(1 + \frac{APR}{m}\right)^{m}. \tag{4.13}$$

Equation (4.13) is the general formula of computing EAR from a known APR. You can also use the previous formula to compute a per-period rate, pp, that is equivalent to an annual rate; simply rearrange the EAR equation to solve for $pp = \frac{APR}{m}$; that is,

$$pp \overset{or}{=} \frac{APR}{m} = (1 + EAR)^{\frac{1}{m}} - 1. \tag{4.14}$$

Below is an example. Notice that in most problems, we care about converting a per-period rate to an effective annual rate or vice-versa. So, usually, we work with pp and EAR.

EXAMPLE 32. *What is the monthly interest rate that is equivalent to 10% interest rate per annum? Here we want to find the per-period rate, i.e., the monthly rate, which is* $pp = \frac{ARP}{m}$, *by knowing that* $EAR = 10\%$. *Straight forward substitution yields*

$$pp = (1 + 0.1)^{\frac{1}{12}} - 1 = 0.0079.$$

Thus earning 0.79 percent interest per month is equivalent to earning 10% annually. What about the annual percentage rate? Well just multiply the per-period by 12 months; that is,

$$pp = \frac{APR}{m} \implies 0.79 = \frac{APR}{12} \implies APR = 0.79 \times 12 = 9.48\%.$$

In this textbook, all problems will state clearly whether the rate is per-period rate, i.e., pp, or an effective annual rate EAR. But, in general, you can easily tell by verifying whether or not the frequency of payment is mentioned. For instance, if a bank is charging and $APR = 10\%$ interest

Period	m	Per period rate $= \frac{APR}{m}$	$EAR = \left(1+\frac{APR}{m}\right)^m - 1$
1 year	1	$\frac{APR}{m} = \frac{8\%}{1} = 8\%$	8.0000%
Semi-annually	2	$\frac{APR}{m} = \frac{8\%}{2} = 4\%$	8.1600%
Quarterly	4	$\frac{APR}{m} = \frac{8\%}{4} = 2\%$	8.2432%
Monthly	12	$\frac{APR}{m} = \frac{8\%}{12} = 0.66\%$	8.2999%
Weekly	52	$\frac{APR}{m} = \frac{8\%}{52} = 0.15\%$	8.3220%
Daily	365	$\frac{APR}{m} = \frac{8\%}{365} = 0.021\%$	8.3277%
Continuously			$e^{APR} = e^{0.08} = 8.3287\%$

TABLE 1. The effective annual rate increases with compounding frequency.

rate, payable quarterly, then the per-period rate is $pp = \frac{APR}{m}$, where $m = 4$; that is, the per-period rate is 2.5% (charged every 3 months). The EAR is

$$\begin{aligned} EAR &= \left(1+\frac{APR}{m}\right)^m - 1 \\ &= \left(1+\frac{10\%}{4}\right)^4 - 1 \\ &= 10.38\%. \end{aligned}$$

Finally, it is worth noting that the higher the frequency of compounding, the higher the EAR. For instance, take an APR of 8% and try to compute the EAR at various compounding periods, e.g., semi-annually, quarterly, monthly, weekly, daily, and continuously. You will notice that the EAR increases with m as demonstrated by Table 1. For continuous compounding, it can be shown that the EAR formula converges, in the limit, to e^{APR}, where $e = 2.718$ is the base of the natural logarithm, or Euler's number.

In Canada, for reasons unclear to me, mortgage interest rates are APRs quoted with semi-annul compounding. Mortgage payments, however, are usually made monthly. For example, if the posted mortgage interest rate is 3.6 percent, this means 3.6 percent compounded semi-annually; that is, the six-month interest rate (or per-period rate) is $pp = \frac{APR}{m} = \frac{3.6}{2} = 1.8$ percent. That being said, to get the appropriate discount rate for a mortgage with monthly payments, you need to find the equivalent monthly interest rate as usual. But, first, you ought to convert the quoted APR rate to its effective annual equivalent using equation (4.13) as follows:

$$EAR = \left(1+\frac{APR}{m}\right)^m - 1.$$

Once EAR is computed, you can then convert this effective annual rate to its monthly equivalent using equation (4.14) as follows:

$$pp = (1+EAR)^{\frac{1}{m}} - 1.$$

Now, let's illustrate this by means of a simple problem.

PROBLEM 12. ***Canadian Mortgages.***[3] *Suppose you took out a \$175,000 Canadian mortgage with a 25-year amortization period, a 5-year term, and a 6 percent posted mortgage interest rate. Compute the monthly payment.*

SOLUTION 12. *In this problem,* $PV = 175000$, CF *is unknown,* $n = 25 \times 12 = 300$ *months, and the interest rate is 6% compounded semi-annually. The effective interest rate is, therefore,*

$$\begin{aligned} EAR &= \left(1 + \frac{APR}{m}\right)^m - 1 \\ &= \left(1 + \frac{0.06}{2}\right)^2 - 1 \\ &= 0.0609. \end{aligned}$$

In other words, 6% compounded semi-annually is equivalent to 6.09% annually. Next, we convert the EAR *to its monthly equivalent. In other words, we want to find the monthly equivalent of 6.09% annually. Using the per-period interest rate formula with* $m = 12$, *we have*

$$\begin{aligned} pp &= (1 + EAR)^{\frac{1}{m}} - 1 \\ &= (1 + 0.0609)^{\frac{1}{12}} - 1 \\ &= 0.004938622. \end{aligned}$$

Technically, the per month interest rate should be 0.005. However, keeping in mind that bankers calculate to 10 decimal places, we wont approximate the per month interest rate. Now, we are ready to compute CF *from the* PV *of an annuity formula:*

$$PV = CF \times \left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right],$$

where $n = 300$ *and* $r = 0.004938622$; *that is,*

$$175000 = CF \times \left[\frac{1}{0.004938622} - \frac{1}{0.004938622(1 + 0.004938622)^{300}}\right].$$

Solving for CF *yields*

$$CF = 1119.7.$$

6. Practice Problems

(1) **PV formula**. You will require \$800 in 5 years. If the rate of return on an alternative investment is 5%, how much will you need to invest today in order to achieve your savings goal?

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[3] This problem is adapted from Brealey et al (2006).

(2) **FV formula**. Suppose you deposited \$1000 into a saving account in bank Alpha. If the bank pays an interest rate of 1.5%, how much will you accumulate in your account in 10 years.

(3) **Number of periods** n. How long will it take for \$500 to grow to \$1,000 at 4 percent interest rate?

(4) **Annuities, AER, and APR.** Retire Well Corp. offers a lifetime annuity to retiring professors. For a payment of \$$x$ at age of 65, the company offers to pay the retiring professor an amount of \$600 per month until death. Professor Charles Xavier is considering this offer. If the remaining life expectancy of Professor X is 20 years and the effective annual rate on this annuity is 6.78% , find x.

(5) **Valuing Delayed Annuities.** Suppose that you will receive annual payments of \$1,000 for a period of 10 years. The first payment will be made in four years from now. If the interest rate is 6%, what is the present value of this stream of payments.

(6) Suppose you took out a \$180,000 Canadian mortgage with a 25-year amortization period, a 5-year term, and a 6 percent posted mortgage interest rate. Compute the monthly payment.

(7) A couple thinking about retirement decide to put aside \$4,000 each year in a savings plan that earns 6 percent interest. How much money will they have accumulated 30 years from now?

(8) Consider the following two plans. Plan A: To receive \$1000 per year for 10 years. Plan B: To receive \$800 per year for 15 years. If interest rate is 5 percent, which plan is better? Explain your reasoning.

(9) Suppose you own a company and you are shopping for a loan of 4000. Your bank is offering 5% rate on loans. Westshore Lending Corp. is offering to lend you \$4000 today. The deal requires that you pay an amount of \$6840 after 2 years. Which deal is better for you?

(10) Suppose your grand grand parent invested \$1000 under your name in the stock market in 1900. If you know that investments in the stock market have increased at an average compounded rate of about 5 percent per year since 1900. How much would that investment be worth in 2008

(11) Your landscaping company can lease a truck for \$8000 a year (paid at year-end) for 6 years. It can buy the truck for \$40,000. The truck will be valueless after six years. (a) If the interest rate your company can earn on its funds is 7 percent, is it cheaper to buy or lease? (b) If the lease payments are an annuity due, is it cheaper to buy or lease?

(12) First National Bank pays 6.2 percent interest compounded semiannually. Second National Bank pays 6 percent interest, compounded quarterly. Which bank offers the higher effective annual rate?

(13) You can buy property today for \$3 million and sell it in 5 years for \$4 million. (You earn no rental income on the property). (a) If the interest rate is 8 percent, what is the present value of the selling price? Is the property investment attractive to you? Why or why not? (b) Would your answer in (a) change if you also could earn \$200,000 per year rent on the property?

(14) If a credit card company quoted you a rate of 2.4 percent a month. You want to borrow \$1000. What is the APR of this loan? What is the EAR of this loan? Based on this EAR, how much you will end up paying in one year?

CHAPTER 5

Bond and Stock Valuation

1. Asset Valuation: A Little Bit of History

The **theory of corporate finance** and **finance theory** are intertwined via many financial notions. Perhaps the key notion that both topics have at their core is **asset valuation**. The analysis of asset valuation has its roots in the **firm-foundation theory**, which asserts that any asset, e.g., common stock, real estate, or even investment project, has a **foundation value** or an **intrinsic value**. Due to market conditions, the actual asset price fluctuates continuously around this intrinsic value; it could fall below or rise above this value. This fluctuation implies that the actual market price of the asset will eventually reach its intrinsic value but will rarely remain at it. This, in turn, creates buying and selling opportunities when the asset is undervalued or overvalued respectively.

Since many people during the early 1930s contributed to the idea of comparing the actual value of an asset to its intrinsic value, the firm-foundation theory was not accredited to one individual. The first formal version of the theory was due to the economist John Burr Williams, who proposed in 1938 a theory of investment value, which postulates that the intrinsic value of an asset can be determined by discounting all its expected future cash flows. In particular, Williams pioneered the dividend valuation of stocks by proposing the rule of present worth, which states that the intrinsic worth of a common stock is the present value of all its future expected dividend payments, i.e., cash flows. Later in 1956, Myron J. Gordon formulated Williams' theory into a valuation equation known as the Gordon growth model. Today, Gordon model is considered the standard model for stock valuation among practitioners in the finance industry. We will discuss stock valuation and Gordon model thoroughly in this chapter.

Surprisingly, the work of Williams (1938) did not get much attention at that time. It did not really become popular until Ben Graham and David Dodd developed their value investing approach and published their book titled "Security Analysis" in 1934. Their idea of value investing is to perform some sort of fundamental analysis that reveals whether or not the asset under consideration is undervalued or overvalued. Their recommendation is to buy the undervalued asset and sell the overvalued one. In this book, Graham suggested to perform fundamental analysis for each individual security by analyzing all the micro variables affecting the firm itself and the industry in which it is operating. Later, in 1949, Graham published a second book; **the Intelligent Investor**, where he refined his own old views and advised to focus more on trying to buy groups of stocks that meet some simple criterion for being undervalued regardless of the industry and with very little attention to the individual company. Graham's value investing view, which has the idea of buying stocks that are traded at less than their intrinsic value at its core, was celebrated by many successful financial practitioners in the industry including Warren Buffett, the chairman of Berskshire Hathaway.[1]

Valuation is a key concept in finance. It is used in finance theory, portfolio management, corporate finance, and capital theory. In a broad sense, the approaches to valuation can be categorized into three general approaches: (1) The **discounted cash flow** (DCF) approach, were the asset under consideration is valued by discounting its future expected cash flows using an appropriate discount rate. (2) The **accounting valuation** or liquidation approach, where the corporation is valued by assessing its existing assets using accounting estimates. (3) The **relative valuation approach**, where the value of an asset is calculated by looking at the value of "comparable" assets relative to a common variable such as earnings or sales. The second and the third approaches are used by accountant in specific circumstances whereas the first approach is the most widely used in business and finance and, for that reason, it will be the focus of this chapter.[2]

The DCF valuation approach treats an asset as a going concern and aims to find the present value by discounting its expected future cash flows using an appropriate discount rate. The word "appropriate" here means a discount rate that reflects the degree of riskiness of the underlying asset. Out of the previously mentioned valuation approaches, the DCF approach is the most comprehensive, well developed, and extensively used among practitioners and academicians in the finance industry. Before we introduce the DCF approach and discuss its variants, below is a short survey of its historical development.

The roots of the DCF approach can be traced back to the development of the rule of present value and interest tables. In an excellent survey on the development of the DCF criteria before 1950, Parker (1968) documents that one of the early interest tables, which was developed back in 1340, is due to Francesco Balducci Pegolotti, who prepared them for Florentine firm of the Bardi in a manuscript that was not published until 1766. The early formal study that laid the basis of the present value rule, which was basically a text in financial mathematics, is due to the mathematician Simon Stevin in a book titled "Tables of Interest." The book was published in Antwerp in 1582. Stevin explained and defined simple and compound interest rates and explained the basis of the present value rule in his textbook.

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[1] Actually this statement was Graham's own words. It was recorded in an interview in March 1976 in Graham's home in California. The interviewer, Mr. Hartman L. Butler, sat for an hour with Graham talking, among many things, about his new book. The interview is known as "An Hour with Mr. Graham."

[2] I should also mention that there is a fourth approach to valuation, which is the contingent claim valuation approach, where the value of an asset is measured using derivative or option pricing models. This approach, however, is a topic by itself and is beyond the scope of this book. For more details on the second and third approaches to valuation, the interested reader is advised to consult Damodaran (2006).

Although, as mentioned in Parker (1968), many contributions to the concept of discounting future cash flows in various applications came from different disciplines such as engineering and actuarial science, it is the work of Marshall (1907), Bohm-Bawerk (1903), Wicksell (1954), and the notable work of Fisher (1907) on capital theory that shaped our understanding of the DCF valuation. Marshall (1907) explained the usefulness of the present value rule in discounting future outflows and inflow of capital. He pointed that when discounting these cash flows, allowance should be made to the risk of failure of the investment capital under consideration. Bohm-Bawerk (1903) used the problem of buying a house for the payment of 20 installments of 1,000 florins each to illustrate the application of net present value. The most notable work on capital theory and DCF valuation is due to Fisher, who published his work in a Book titled "The Rate of Interest" in 1907. The book was extensively revised in 1930 and published under the title "The Theory of Interest." In his books, Fisher suggested four ways of choosing between investment alternatives. The **net present value** (**NPV**) criterion and the **internal rate of return** (**IRR**) criterion were among the four suggested alternatives. Nowadays, both approaches are considered standard practices and routinely used among practitioners in the finance industry. The NPV of any stream of cash flow is the discounted value of future cash inflows net any cash outflows. The IRR of any investment project is simply the rate that equates today's value with the present value of all future cash flows. We will explore both criterion thoroughly in this chapter.

Although Fisher discussed a variant of the IRR, he did not derive it thoroughly. Boulding (1935) derived the IRR for a single investment/asset/firm by equating the initial investment cost to the present value of the expected cash flows of the single investment. Boulding pointed out that the IRR of a single investment could be calculated from the following equation

$$V_0 = \frac{E\left[NCF_1\right]}{(1+IRR)} + \frac{E\left[NCF_2\right]}{(1+IRR)^2} + \cdots + \frac{E\left[NCF_n\right]}{(1+IRR)^n}, \tag{5.1}$$

where V_0 is the initial investment or the current value of the enterprise, $E\left[NCF_t\right]$ is period t's expected **net cash flow** (positive or negative) for $t = 1, 2, \ldots, n$, where n is the total number of periods, and IRR is the internal rate of return or the discount rate. Boulding argued that any rational investor should choose the project that maximizes his or her IRR. Solving for IRR in equation (5.1) is not an easy task. In fact, the IRR is not unique. The solution yields multiple rates of return, which could be ambiguous and confusing sometimes. Boulding recognized that the solution of the previous equation is not mathematically simple and, mistakenly, thought that a single solution could be found. He did not regard the existence of multiple internal rates of return as problematic.

Keynes (1936) gave the, now familiar and widely known, definition of IRR as the rate of return that sets the current asset price equal to the present value of all its future returns. Keyens referred to this definition of IRR as the **marginal efficiency of capital**. This is the definition that we will entertain in this chapter when discussing the IRR method.

Samuelson (1937) discussed the contributions made by Keynes (1936) and Boulding (1935) and noted that Boulding's internal rate of return and Keynes' marginal efficiency of capital are equivalent. Samuelson, however, rejected Boulding's view of maximizing IRR. Samuelson pointed out the possibility of no rate of return or of multiple rates, which is considered a major drawback of the IRR criterion.

Interest in DCF methods started to become more significant in the 1950s and became widespread in the 1960s. Parker (1968) reviewed the history of the DCF methods and their use in the 1950s and the 1960s. Nowadays, DCF methods are extensively used in financial valuations as well as a wide variety of applications in finance, business, and economic.

Asset valuation is a method of computing how much an asset is worth today, or in other words, how much investors are willing to pay for a particular asset today. This requires discounting the expected future **risky** cash flows generated by the asset over its future lifetime. Notice that the two key elements of asset valuation are the **lifetime** of the asset and the anticipated **risk** in its expected cash flows. This is not surprising since time and risk are the two key ingredients of making any investment decision. In the absence of risk, either in the cash flows or the discount rate, the present value, V_0, of any asset today is the sum of all its certain future cash flows discounted at the risk-free discount rate; that is,

$$V_0 = \frac{CF_1}{(1+rf_1)} + \frac{CF_2}{(1+rf_2)^2} + \cdots + \frac{CF_n}{(1+rf_n)^n},$$

where CF_t is the asset's certain cash flow in period t, for $t = 1, 2, \ldots, n$, rf_t is the risk-free rate of return in period t, and n is the lifetime of the asset.

This way of asset valuation with **certainty** is straightforward; just add the future cash flows after discounting them by the period's risk-free rate of return. The valuation problem is more complicated when the future cash flows are subject to random events, i.e., the debt issuer's ability to pay the interest payment in case of bonds, the equity issuer's ability to pay dividends in case of stocks, and the project's ability to generate revenue in case of commercial projects. Thus, solvency, financial strength, and commercial success are considered random events that affect the future cash flows in case of bonds, stocks, and commercial projects respectively. Since these random events are in play, their associated outcomes, i.e., their associated cash flows, are considered random variables with probabilistic realizations. Hence, their expected values are what should be considered in their valuations. If we let CF_t denote the **random variable** representing the cash flow of a given asset at time t and everything else is the same, then the value of this asset today must follow the following net present value formula.

$$V_0 = \frac{E\left[CF_1\right]}{(1+rf_1)} + \frac{E\left[CF_2\right]}{(1+rf_2)^2} + \cdots + \frac{E\left[CF_n\right]}{(1+rf_n)^n}. \tag{5.2}$$

The formula in (5.2) is the reference formula for any DCF model. Since randomness implies uncertainty, the only thing remaining is to incorporate such uncertainty/risk in the previous formula. The introduction of uncertainty in equation (5.2) produced many variations of the previous basic DCF formula. One approach augments the risk-free rate by a premium, which takes into account the risk associated with the future uncertainty of the expected cash flows. This risk premium varies from one asset to another. The adjustment of the discount rate in this way produces the following variation of the basic DCF formula:

$$V_0 = \frac{E\left[CF_1\right]}{(1+rf_1^*)} + \frac{E\left[CF_2\right]}{(1+rf_2^*)^2} + \cdots + \frac{E\left[CF_n\right]}{(1+rf_n^*)^n}, \tag{5.3}$$

where $rf_t^* = rf_t + \pi$ is the **risk-adjusted discount rate** at time t and π is a fixed discount factor. An alternative approach of incorporating uncertainty in the basic DCF formula is to keep the risk-free rate as the discount rate but to decrease period t's cash flow by a factor γ_t that represents some risk or insurance premium. This factor takes into account the riskiness of the future cash flow. The adjustment of the cash flows in this way produces the following variation of the basic DCF formula:

$$V_0 = \frac{E\left[CF_1^*\right]}{(1+rf_1)} + \frac{E\left[CF_2^*\right]}{(1+rf_2)^2} + \cdots + \frac{E\left[CF_n^*\right]}{(1+rf_n)^n}, \tag{5.4}$$

where $E\left[CF_t^*\right] = E\left[CF_t\right] - \gamma_t$ is the **risk-adjusted cash flow** at time t. A third approach in the same spirit of the second is to also keep the risk-free rate as the discount rate but distort the probability distribution of future cash flows to incorporate risk or uncertainty, and then take expectations of the modified cash flows. The notion of uncertainty can also be captured by projecting the cash flows of the asset over different states of nature and then finding a price of the asset today conditional on the realized state in the future.

Among the previous four approaches, the first two are the most frequently used among practitioners in the industry. The third and fourth approaches are more methodological and technical in nature and are based on certain restrictive assumption, which makes their implementation difficult in practice. But, regardless whether the approach is easy or difficult to implement, it should be noted that the success of the discounting approach in case of uncertainty requires the computation of risk premia. The way risk is introduced in the basic DCF formula, which resulted in the previous variations of the DCF model, is guided by financial theories. In particular, the first two variations are made possible using financial theories on modeling risk-reward relations, e.g., CAPM and other financial models.

2. Bond Valuation

2.1. PV of a Bond. A corporate bond is a debt instrument issued by corporations to raise funds. It is an agreement between the issuer, i.e., the corporation, and the holder, i.e., the saver or investor, by which the issuer is obligated to pay a fixed amount every specific period of time till maturity, which we will denote by n. The fixed amount that is paid every specific period of time is known as the **coupon payment**, which we will denote by C. The coupon payment is simply the **fixed** interest payment paid to the bondholder. At maturity, the debt is repaid: the borrower pays the bondholder the bond's face value, denoted by F, which is also known as **principal**, **par value**, or **maturity value**. The interest payment or coupon payment is calculated by multiplying a fixed rate, known as the **coupon rate**, by the face value. The coupon rate, denoted by the letter small c, is always expressed as an **annual** interest rate. Thus, the coupon payment is simply calculated by multiplying the fixed coupon rate times the bond's face value:

$$C = c \times F. \tag{5.5}$$

For instance, if the bond's face value $F = \$1000$ and the coupon rate $c = 10\%$, then this bond pays \$100 every year till maturity. Sometimes, the payments are paid semiannually, quarterly, or even monthly. In the previous example, if the coupon payments are paid semiannually, then the bondholder is entitled to receive two payments of \$50 each in a year till maturity. If the bond matures in 5 years. Then, the bondholder is entitled to 10 payments of \$50 each. In other words, multiply n by 2 and divide C by 2 for semi-annual payments. For quarterly payments, we multiply and divide by 4, and for monthly payment, we multiply and divide by 12.

It is worth noting that the face value of any bond remains **unchanged**. However, due to demand and supply conditions, the bond is traded a price that could be above or below the face value. It is essential to keep in mind that the face value of the bond is not the bond price. The former is fixed whereas the latter fluctuates due to speculative behavior of market participants. We will denote the bond price by B.

$$\begin{array}{ccccccccc}
 & & C & & C & & C & & C+F \\
 & & \uparrow & & \uparrow & & \uparrow & \cdots & \uparrow \\
0 & \_\_\_\_ & 1 & \_\_\_\_ & 2 & \_\_\_\_ & 3 & \_\_\_\_ & n \\
\downarrow & & & & & & & & \\
F & & & & & & & &
\end{array}$$

A bond is valued by discounting all future payments plus the face value, which is paid at maturity, using the prevailing discount rate, which is denoted by r. The above time line illustrates these future payments. An arrow pointing up implies a cash inflow and an arrow pointing down implies an outflow. Notice that in year 0 the bondholder pays the face value. Every year after that, the holder receives a fixed payment C every period till maturity, where the face value is finally paid back. Thus, the present value of the bond is the present value of all coupon payments plus the present value of its face value at maturity; that is,

$$B \overset{or}{=} PV(bond) = PV(coupon\ payments) + PV(face\ value),$$

or equivalently,

$$\begin{aligned}
(5.6)B &= \underbrace{\frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n}}_{PV(coupon\ payments)} + \underbrace{\frac{F}{(1+r)^n}}_{PV(face\ value)} \\
&= C\left[\left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^2 + \cdots + \left(\frac{1}{1+r}\right)^n\right] + F\left(\frac{1}{1+r}\right)^n
\end{aligned}$$

Notice that the first n terms in equation (5.6) is just an annuity and the last term is just a single amount that is due at year n. Thus, using equation (4.10), and the standard discounting formula, we have

$$B \overset{or}{=} PV(bond) = C \times \left[\frac{1}{r} - \frac{1}{r\,(1+r)^n}\right] + F\left(\frac{1}{1+r}\right)^n. \tag{5.7}$$

Equation (5.7) is the standard valuation formula for any bond. Here is an example.

EXAMPLE 33. *Let's compute the present value of a 6-year bond with a face value of \$1000 and 10 percent coupon rate, using a discount rate of 12 percent. In this case* $F = 1000$, $c = 10\%$, $n = 6$, *and* $r = 12\%$. *Then,*

$$C = c \times F = 0.1 \times 1000 = 100$$

and

$$\begin{aligned}
B \overset{or}{=} PV(bond) &= \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^6} + \frac{F}{(1+r)^6} \\
&= \frac{100}{(1+0.12)^1} + \frac{100}{(1+0.12)^2} + \cdots + \frac{100}{(1+0.12)^6} + \frac{1000}{(1+0.12)^6} \\
&= 100 \times \left[\left(\frac{1}{1+0.12} \right)^1 + \left(\frac{1}{1+0.12} \right)^2 + \cdots + \left(\frac{1}{1+0.12} \right)^6 \right] \\
&\quad +1000 \left(\frac{1}{1+0.12} \right)^6 \\
&= 100 \times \left(\frac{1}{0.12} - \frac{1}{0.12\,(1+0.12)^6} \right) + 1000 \left(\frac{1}{1+0.12} \right)^6 \\
&= 917.1
\end{aligned}$$

REMARK 8. *In the previous example, if coupon payments are semiannually, then we ought to convert the entire problem to reflect the new period adjustments. In particular, we need to adjust three things: (1) the number of periods, (2) the coupon payment, and (3) the discount rate. The number of periods should be the number of years* $n = 6$ *times the frequency* $m = 2$ *(semiannual). Thus, on the time line, we ought to have 12 periods. The coupon payment* C *is paid twice a year. Thus, the per-period payment is* $\frac{C}{m} = \frac{100}{2} = 50$. *Finally, the discount rate* r *is an annual discount rate. We cannot use an annual discount rate to discount 6-month payments. So, we need to convert the annual discount rate to a per-period discount rate. To this end, we use equation (4.14)*

$$r^* = pp = (1 + EAR)^{\frac{1}{m}} - 1,$$

where r^* *is the semi-annual (or per-period) rate,* $m = 2$ *is the frequency, and* $EAR = 12\%$. *Thus, the per-period rate is*

$$r^* = (1 + 0.12)^{\frac{1}{2}} - 1 = 5.8\%.$$

The present value equation after taking into account all the above per-period adjustments is

$$\begin{aligned}
B \overset{or}{=} PV(bond) &= \frac{C/2}{(1+r^*)^1} + \frac{C/2}{(1+r^*)^2} + \cdots + \frac{C/2}{(1+r^*)^{12}} + \frac{F}{(1+r^*)^{12}} \\
&= \frac{50}{(1+r^*)^1} + \frac{50}{(1+r^*)^2} + \cdots + \frac{50}{(1+r^*)^{12}} + \frac{1000}{(1+r^*)^{12}}.
\end{aligned}$$

CONCLUSION 1. *In general for any frequency* m *and per-period rate* r^*, *the bond present value is subject to the following transformation:*

(5.8)

$$B \overset{or}{=} PV(bond) = \frac{C/m}{(1+r^*)^1} + \frac{C/m}{(1+r^*)^2} + \cdots + \frac{C/m}{(1+r^*)^{n\times m}} + \frac{F}{(1+r^*)^{n\times m}},$$

or equivalently,

$$B \overset{or}{=} PV(bond) = \frac{C}{m} \times \left[\frac{1}{r^*} - \frac{1}{r^*\left(1+r^*\right)^{n\times m}}\right] + F\left(\frac{1}{1+r^*}\right)^{n\times m}. \tag{5.9}$$

3. Cost of Debt & Yield to Maturity

Depending on how you look at it, a bond could be a plan to raise fund or it could be an investment plan. From the firm's point view, issuing a bond is a borrowing plan via a debt instrument. From the investor's point view, however, it is an investment that achieves a rate of return. In this section, we seek to answer the following question: How to compute the rate of return or the yield on a bond if it is considered an investment plan? And how to compute its cost as a debt instrument? We begin by answering the first question.

The bond holder receives C every period and the traded value of his or her investment is B. Thus, the rate of return every period is simply obtained by dividing the coupon payment C over the bond price B. For instance, if the bond holder receives \$50 per year for n years and the current bond price $B = \$500$. Then, the holder achieves every period a rate of return on investment, or a yield, of 10%. This yield is known as the **current yield.**

DEFINITION 33. *The **current per-period yield** is the per-period coupon payment divided by bond price.*

$$\text{current per-period yield } = \frac{C/m}{B}, \tag{5.10}$$

where m is the frequency.

REMARK 9. *For bonds that pay C annually, $m = 1$, and the current yield is just C/B.*

In addition to the current per-period yield, the bond holder may or may not sell the bond. If the bond is sold at a premium, i.e., at a price above the face value, then the investor achieves a profit equals to the difference between the sale price and the buy price. Conversely, if it is sold at a discount, then the investor achieves a loss. The capital gain/loss realized from the sale of an existing bond is computed in the form of a rate by dividing the price change, i.e., sale price minus initial buy price, over the bond's initial buy price.

DEFINITION 34. *The capital gain/loss from the sale of a bond is the change in price divided by the bond price; that is,*

$$\text{capital gain } = \frac{\text{price change}}{B}. \tag{5.11}$$

The rate of return at any period of time is, therefore, the sum of the current yield and the capital gain.

DEFINITION 35. *The rate of return is the total income per period per dollar invested. It is the current per-period yield plus capital gain:*

$$\text{Rate of return} = \frac{C/m + \text{price change}}{B}. \tag{5.12}$$

For instance, in the previous example, it the bond was sold at \$600, then the rate of return realized in this period is

$$\frac{50 + (600 - 500)}{500} = 30\%.$$

This 30% is simply the 10% current yield plus 20% capital gain.

Granted that the bond holder realizes a current yield of $\frac{C/m}{B}$ every period (plus capital gain if he or she chooses to sell the bond), the rate of return as a measure doesn't take into account the change in the bond's value over its life time. We need a measure of return that takes into account both current per-period yield and change in bond value over time. This measure is known as the **yield to maturity** or **YTM** for short, and is defined as follows.

DEFINITION 36. *The **yield to maturity** (**YTM**) is the discount rate that equates the bond's price to the present value of all its future payments; that is, YTM is such that*

$$B = \underbrace{\frac{C}{(1+YTM)^1} + \frac{C}{(1+YTM)^2} + \cdots + \frac{C}{(1+YTM)^n} + \frac{F}{(1+YTM)^n}}_{PV(bond)} \tag{5.13}$$

The discount rate in equation (5.13) is the rate that equates the value of the bond today to the present value of all its future payments. It is the rate of return generated over the life time of the instrument. Here is a simple example to illustrate this concept.

EXAMPLE 34. *Consider a 2-year bond with face value $F = 1000$ and a coupon rate $c = 10\%$. The bond has been traded in the secondary market for a year now and the current prevailing price is \$900. You are expecting the price of the bond to increase and, based on your expectation, you are contemplating the decision to buy the bond today. What is the rate of return on this investment plan? In other words, what is the YTM should you decide to invest in such a bond? First, notice that if you decide to buy the bond today, then you will only receive one coupon payment of \$100 and the face value $F = 1000$ in one year when the bond matures. The YTM is obtained by solving for the discount rate that equates the value of the bond today to* *the present value of all its future payments; that is, the return is obtained by solving the following equation for YTM*

$$900 = \frac{1100}{(1+YTM)}.$$

This gives a $YTM = 22.2\%$.

The previous example was simple enough to solve for the YTM. Things could easily get complicated if more terms are introduced on the right-hand-side of the YTM equation. For instance, if the bond in the pervious example is a three year bond, then the YTM equation becomes

$$900 = \frac{100}{(1+YTM)} + \frac{1100}{(1+YTM)^2} \tag{5.14}$$

How can we compute the YTM in this case? Trying to solve for YTM in the previous equation is not an easy task. The only **general** procedure for calculating the YTM is trial and error. In practice, some investors use financial calculators or spreadsheets to do so. However, trial and error remains the most convenient and widely used method to compute the YTM. The trial and error method works as follows: You guess an interest rate and compute the PV of the bonds's payment (the right-hand-side of the YTM equation). If the PV is greater than the bond price, the discount rate must have been too low. You should then try a higher interest rate and vice-versa. Keep repeating your guess until the value of the right-hand-side of the YTM equation is very close to the bond price (the value on the left-hand-side). To see this, let's try 15 percent. The value of the right-hand-side of equation (5.14) when $YTM = 0.15$ is 918.71, which is greater than 900:

$$\frac{100}{(1+0.15)} + \frac{1100}{(1+0.15)^2} = 918.71 > 900.$$

So, we should try a slightly higher rate; say $YTM = 0.16$. Then,

$$\frac{100}{(1+0.16)} + \frac{1100}{(1+0.16)^2} = 903.69 > 900.$$

We are still slightly above 900. So, we should try $YTM = 0.17$. This gives

$$\frac{100}{(1+0.17)} + \frac{1100}{(1+0.17)^2} = 889.03 < 900.$$

Thus, it must be somewhere between 0.16 and 0.17. We can use data interpolation at this point or we can continue to guess. Using a spreadsheet, guessing is usually easier. So, let's try $YTM = 16.2\%$. This gives

$$\frac{100}{(1+0.162)} + \frac{1100}{(1+0.162)^2} = 900.7 \approx 900.$$

The exact YTM is actually 16.24. But, our guess of 16.2 is good enough.

REMARK 10. *If the coupon payments are paid annually, the YTM is an annual yield. However, for different frequencies, the YTM is the per-period yield, which we will denote by YTM^*. In this case, you will need to convert this per-period rate to its equivalent annual rate to obtain the YTM. Here is an example.*

EXAMPLE 35. [3]*Consider a three-year bond with $F = 1000$ and $C = \$55$ paid semi-annually. Suppose the bond is traded at a price $B = 1024.694$. What is the YTM on that bond? Notice here that we have 6 payments of \$27.5 each. The YTM equation is*

$$1024.694 = \frac{27.5}{(1+YTM^*)} + \frac{27.5}{(1+YTM^*)^2} + \cdots + \frac{1027.5}{(1+YTM^*)^6}.$$

Notice also that, since we converted all payments to per-period, the YTM^* is the per-period yield. You can verify that $YTM^* = 2.3047\%$. You can do so using a financial calculator or by trial and error using a spreadsheet. If you are using a financial calculator, use the following inputs: $n = 6$ (semi-annual periods), $PV = -1,024.694$ (this is the price or PV input in a financial calculator), $FV = 1000$ (the face value is FV input), and finally the coupon payment input is $PMT = 27.50$. The resulting yield is of course a six-month rate and ***not*** *an annual rate. You can annualize it, using simple interest, by doubling it as $2.3047\% \times 2 = 4.6094\% \approx 4.61\%$. This is how bond dealers usually quote the YTM. Alternatively, if you want to account for compound interest rate, you can compute the effective annual yield using the EAR formula in equation (4.14):*

$$\begin{aligned} EAR &= (1+YTM^*)^m - 1 \\ &= \left(1+\frac{0.046094}{2}\right)^2 - 1 \\ &= 4.66\%, \end{aligned}$$

where $m = 2$ is the frequency and $YTM^* = pp = \frac{0.046094}{2}$ is the per-period yield.

REMARK 11. *As mentioned above, finding an exact YTM is not an easy task; you can use the trial and error method that we outlined in the examples above, or alternatively, you can compute the YTM from the following approximation:*

$$YTM \approx \frac{C/m + \frac{F-B}{n\times m}}{\frac{F+B}{2}}, \tag{5.15}$$

where everything is defined as above. For instance, in the previous example, the YTM from the approximation formula is

$$YTM \approx \frac{27.5 + \frac{1000-1024.694}{6}}{\frac{1000+1024.694}{2}} = 2.399\%.$$

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[3] Adapted from Bealey et al (2006).

This is close enough to 2.34%. The approximation formula is a quick and easy way to find the YTM in the absence of a financial calculator or a spreadsheet.

PROBLEM 13. *Consider a bond with face value* $F = \$1000$, *coupon rate* $c = 6\%$ *paid semi-annually, and maturity of 2 years. Suppose that the bond is immediately traded after issuance at a price* $B_0 = 800$, *i.e., the bond is traded at a discount. What is the yield to maturity if you want to buy the bond today?*

SOLUTION 13. *Here,* $F = \$1000$,

$$C = c \times F = 0.06 \times 1000 = 60 \ (\textit{paid semi-annually}).$$

Let the frequency be $m = 2$ *(semi-annually). Then, the per-period payment is* $C/m = 60/2 = 30$. *Notice the distinction here between the frequency,* $m = 2$ *(semi-annually) and the maturity* $n = 2$ *years. Therefore, on the time line, we have actually* $n \times m = 2 \times 2 = 4$ *periods and, in each period, there is a payment of 30 and the face value at the last period. Finally, we have* $B_0 = 800$. *The yield to maturity is the rate that equates today's value to the present value of all future payments. It is calculated from the following equation*

$$800 = \frac{30}{(1+YTM^*)^1} + \frac{30}{(1+YTM^*)^2} + \frac{30}{(1+YTM^*)^3} \frac{30+1000}{(1+YTM^*)^4}.$$

Notice, however, that this YTM is the not the annual yield; rather it is the semi-annual (or per-period) yield to maturity. This is true since we converted the problem into per-period durations. For that reason, the YTM is tagged with a "" in the previous equation to remind the reader that we are computing a semi-annual rate. Using the approximation method, we have*

$$YTM^* \approx \frac{\frac{C}{m} + \frac{F-B_0}{m \times n}}{\frac{F+B_0}{2}} = \frac{30 + \frac{1000-800}{4}}{\frac{1000+800}{2}} = 0.0889 \ (\textit{or } 8.89\%);$$

that is, the per-period yield to maturity or the semi-annual yield to maturity is $YTM^* = 8.89\%$. *We are not done yet. When asked to compute the YTM, we should always report the annual yield. In this case, since we have the semi-annual yield, we ought to convert it to the equivalent annual yield using the conventional EAR formula, where* $APR/m = pp = YTM^* = 8,89\% = 0.0889$. *This gives*

$$\begin{aligned} YTM &= (1+YTM^*)^m - 1 \\ &= (1+0.0889)^2 - 1 = 0.1857 \end{aligned}$$

Thus, the YTM is 18.57%.

So far, we discussed the bond's rate of return over a particular investment period and its yield to maturity. It is important to make the clear distinction between both concepts. The yield to maturity is the discount rate that equates the bond price B to the present value of all its future payments. It is, thus, a measure of the average rate of return over the bond's life if the holder held it till maturity. In contrast, the rate of return can be calculated at any particular period. It is simply the current yield and the capital gain/loss over that period and not the entire lifetime of the bond.

At any rate, whether the return is calculated in one period or over the life time, the bond so far was analyzed as an investment plan from the holder's perspective. What about the other view? When bond is considered a borrowing plan from the firm's perspective, the cost of debt, which we will denote by k_b for reasons that will become clear shortly, is computed in the exact same way as the YTM. It is the rate that equate the value of the bond today to the present value of all its future payments. The only difference is that this rate is a **cost** for the firm rather than a return. Thus, k_b is such that

$$B = \frac{C}{(1+k_b)^1} + \frac{C}{(1+k_b)^2} + \cdots + \frac{C}{(1+k_b)^n} + \frac{F}{(1+k_b)^n}, \tag{5.16}$$

and is computed by trial and error or by the approximation formula:

$$k_b \approx \frac{C + \frac{F-B}{n}}{\frac{F+B}{2}}. \tag{5.17}$$

3.1. Bond Market Data. Bonds are traded usually at prices above or below the par or face value. For instance, a bond with a face value $F = \$1000$ could be traded at a price $B = 1100$ or at a price $B = 950$. In the former case, where the trading price is above the face value, the bond is said to be traded at a premium. In the latter case, where the trading price is below the par value, the bond is said to be traded at a discount. Of course, there is the possibility that the bond is traded at a price equal to its face value. In such a case, the bond is said to be traded at par. We will explore the, interesting and fundamental, relationship between bond prices and interest rates in the following section. In this section, we focus on how to read and understand the bond data that are routinely reported on financial Web sites or major newspapers.

The reported prices are the closing prices as of 4 pm and are always quoted for the previous day's trading activities. The **bid** price is the price you receive if you **sell** the bond to a broker or a bond dealer. The prices are quoted as percentages of face value. So, for instance, for a bond with a face value $F = 1000$, a quoted price of 103.5 means 103.52 percent of the \$1000 face value; that is,

$$\frac{103.52}{100} \times 1000 = \$1035.2.$$

The **asked** price is the price you pay if you buy the bond. For the same bond, for instance, an asked price of \$102.52 means you have to pay \$1025.2

should you decide to buy it. The difference between the bid and ask prices, known as the **bond spread**, is simply the profit of the bond dealer.

If you happen to buy the bond between coupon payments, you pay more than the quoted price. The reason is that you need to compensate the seller for the coupon interest earned from the tie of the last coupon payment to the purchase (settlement) date. This amount is known as **accrued interest** and is computed as follows:

$$\text{Accrued interest} = C \times \frac{\text{number of days from last coupon to purchase date}}{\text{number of days in coupon period}}.$$

EXAMPLE 36. [4] ***What you really pay when you buy a bond.*** *Suppose today is August 14, 2010, and the quoted price of a Canada 6 percent bond with face value $F = \$1000$ maturing June 1, 2012, is 107.7. The bond pays semi-annual interest. The last \$30 coupon payment was paid on June 1, 2010, and the next coupon payment will be paid on December 1, 2010. The number of days from the last coupon payment to the purchase date is 74 (29 days from June 1 to June 30, 31 days in July, and 14 days in August) and the total number of days in the coupon period is 183 (from June 1 to December 1). The accrued interest is*

$$\$30 \times \frac{74}{183} = \$12.13.$$

Thus, the total cost of buying one bond is

$$\$1077 + \$12.13 = \$1089.13$$

4. Interest Rate and Bond Prices

To understand the relation between interest rate and bond prices, we should first make the clear distinction between coupon rate and interest rate. The coupon rate, c, is a fixed rate used to compute the coupon payment on a bond. It doesn't change or fluctuate and doesn't respond to market conditions. The market interest rate, r, is the **required rate of return** or **expected rate of return** by investors. It is the return on any asset in general that an investor would require to invest in such asset.[5] Suppose that you saved \$1000 and you want to achieve some future gain on your savings. In chapter 1, we mentioned that you have two choices: either to invest your money directly in a bond or a stock, i.e., in the financial market, or indirectly using the service of a financial intermediary. Suppose the bank pays a market rate $r = 10\%$ on deposits whereas an equally risky three-year bond with face value $F = \$1000$ pays a coupon rate $c = 10\%$. Then, you will be indifferent between putting your money in the bank or investing in a bond since both alternatives will yield a rate of return of 10%. Notice here that the bond will be traded at par value. To see this, simply find the bond price by discounting all future payments using the required rate of return $r = 10\%$ as the discount rate. This gives

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[4] Example 5.1, page 133, in Brealey et all (2006).

[5] In finance, the terms **required return** and **expected return** mean the same thing and are often used interchangeably.

$$\begin{aligned} B &= PV \text{ at } 10\% = \frac{100}{(1+0.1)^1} + \frac{100}{(1+0.1)^2} + \frac{1100}{(1+0.1)^3} \\ &= 1000. \end{aligned}$$

Thus, when the coupon rate is equal to the market interest rate, the bond is traded at par:

$$c = r \implies B = F.$$

But, what will happen to the bond price if the market rate is higher than the coupon rate? To answer this question, we need to understand the behavior of market participants in relation to interest rate movements. This behavior is known as **speculation**.

Economic theory tells us that there are three motives behind the demand for money: People demand money to **undertake transactions**, or for **precaution**, or for **speculation**. The first motive is obvious: people demand money to buy goods and services. The second motive is also obvious: people tend to set aside some money to meet any unexpected events. The third motive is not so obvious. Speculation involves trading a financial instrument involving high risk in **expectation** of significant returns. The key word here is "expectation." When investors expect the market interest to decrease, then bond markets are more appealing than banks. This speculation of low interest rate will, thus, result in an increase in the demand for money. Investors will simply withdraw their bank deposits and purchase/invest in bonds. The demand for bonds will increase, and in turn, causing the bond price to rise. Thus, as market interest rate goes down, speculation activities will result in an increase in bond prices. Conversely, if investors expect the interest rate to rise, they will demand less money to speculate, i.e., they will leave their money at the bank since this plan is more rewarding than investing in bonds. The demand for bonds will, thus, decrease pushing bond prices down. For instance, in the example above, if the market interest rate is expected to increase to $r = 15\%$ say, then you would require a rate of return of 15% to invest in bonds. But the coupon rate $c = 10\%$. Thus, you would rather keep your money in the bank. It is sensible to assume that every other investor will follow the same rule, and consequently, we expect that the demand for bonds will drop causing the bond price to drop as well. This implies that if the interest rate is expected to rise above the coupon rate, the bond ought to sell at a price less than the face value. This drop in price is due to market speculation. To see this, simply discount the future payments at the higher required rate of return of 15%:

$$\begin{aligned} B &= PV \text{ at } 15\% = \frac{100}{(1+0.15)^1} + \frac{100}{(1+0.15)^2} + \frac{1100}{(1+0.15)^3} \\ &= 885.84 < F = 1000. \end{aligned}$$

Thus, when

$$c < r \implies B < F.$$

Similarly, when

$$c > r \implies B > F$$

and the bond sells for more than the face value. The previous relation is a key bond pricing rule: when interest rate is expected to rise, bond prices will fall. Conversely, when interest rate is expected to decline, bond prices will increase. The reason behind this negative relationship is market speculation as explained above.

CRITERION 1. ***Bond pricing rule.*** *When the market interest rate exceeds the coupon rate, bonds are traded at discount, i.e., they sell at a price less than face value. When the market interest rate is below the coupon rate, bonds are traded at a premium, i.e., they sell at a price more than face value. Finally, when the market rate equals the coupon rate, bonds are traded at par.*

REMARK 12. *In the previous sections, we introduced a variety of rates of return; namely, rate of return on a bond, yield to maturity on a bond, YTM, discount rate of a bond, r, and cost of debt k_b. The reader should be clear about the differences between these rates. First, these are all rates of returns, i.e., in percentages rather than monetary values. Second, the rate of return and YTM are two alternatives ways to compute the rate of return on a bond from the investor point of view. The YTM is better since it takes into account all future bond payments, i.e., it takes into account the time value of money. The discount rate is a rate used to discount all future coupon payments and the face value pertaining to a bond to arrive at the bond price, which is the present value. Investors use discount rates to assess how much should a bond be worth today. The discount rate is usually taken to be the market rate or the interest rate on deposits offered by the bank. The reason of choosing the discount rate for a bond to be equal to the bank rate follows from the fundamental relation between bond prices and market rates that we explored in this section. Finally, the cost of debt is a rate that is equivalent to the YTM but from the corporation's perspective. It is the cost of debt that takes into account all future cash outflows that the firm ought to pay on the bond as a debt instrument.*

5. Some Practice Problems on Bond Valuation

PROBLEM 14. *Consider a bond with face value $F = \$1000$, coupon rate $c = 8\%$, and maturity of 2 years. Suppose that the bond is immediately* *traded after issuance at a price* $B_0 = 800$, *i.e., the bond is traded at a discount. What is the yield to maturity if you want to buy the bond today?*

SOLUTION 14. *First, notice the following:*$F = \$1000$,

$$C = c \times F = 0.08 \times 1000 = 80,$$

the frequency, $m = 1$ *(annually), the maturity,* $n = 2$ *years, and* $B_0 = 800$. *The yield to maturity is the rate that equates today's value to the present value of all future payments. It is calculated from the following equation*

$$B_0 = \frac{C}{(1+YTM)^1} + \frac{C+F}{(1+YTM)^2};$$

that is,

$$800 = \frac{80}{(1+YTM)^1} + \frac{1080}{(1+YTM)^2}$$

Using the approximation method:

$$YTM \approx \frac{C + \frac{F-B_0}{2}}{\frac{F+B_0}{2}} = \frac{80 + \frac{1000-800}{2}}{\frac{1000+800}{2}} = 20\%.$$

PROBLEM 15. *Consider a bond with face value* $F = \$1000$, *coupon rate* $c = 8\%$ *paid semi-annually, and maturity of 2 years. Suppose that the bond is immediately traded after issuance at a price* $B_0 = 800$, *i.e., the bond is traded at a discount. What is the yield to maturity if you want to buy the bond today?*

SOLUTION 15. *This problem is similar to Problem 13. Here, we have* $F = \$1000$,

$$C = c \times F = 0.08 \times 1000 = 80 \text{ (paid semi-annually)};$$

$m = 2$ *and* $n = 2$. *Then, the per-period payment is* $C/m = 80/2 = 40$. *On the time line, we have* $n \times m = 2 \times 2 = 4$ *periods and, in each period, there is a payment of 40 and the face value at the last period. Finally, we have* $B_0 = 800$. *The per-period yield to maturity is calculated from the equation:*

$$800 = \frac{40}{(1+YTM^*)^1} + \frac{40}{(1+YTM^*)^2} + \frac{40}{(1+YTM^*)^3} \frac{40+1000}{(1+YTM^*)^4}.$$

Using the approximation method, we have

$$YTM^* \approx \frac{\frac{C}{m} + \frac{F-B_0}{m\times n}}{\frac{F+B_0}{2}} = \frac{40 + \frac{1000-800}{4}}{\frac{1000+800}{2}} = 0.10;$$

that is, the per-period yield to maturity or the semi-annual yield to maturity is $YTM^* = 10\%$. *Finally, we compute the equivalent annual yield using the conventional* EAR *formula as follows:*

$$\begin{aligned} YTM &= (1+YTM^*)^m - 1 \\ &= (1+0.10)^2 - 1 = 0.21 \end{aligned}$$

Thus, the YTM is 21%.

PROBLEM 16. *Consider a bond with face value* $F = \$1000$, *coupon rate* $c = 8\%$ *paid semi-annually, and maturity of 2 years. Suppose that you are thinking of buying the bond at the end of year 1 and you are planning to hold the bond till maturity. If the bank rate is 4%, how much you should pay for that bond?*

SOLUTION 16. *Here, you need to find the fundamental price of the bond or the present value of the bond. You need first, however, to convert every thing to per-period. We have* $F = \$1000$, $C = c \times F = 0.08 \times 1000 = 80$ *(paid semi-annually). Thus, the per-period payment is* $C/m = 80/2 = 40$. *The frequency,* $m = 2$ *(semi-annually), maturity,* $n = 2$ *years, and thus, on the time line, we have* $n \times m = 2 \times 2 = 4$ *periods and, in each period, there is a payment of 40. Finally, the face value is paid at the fourth period. The bond is purchased at end of year 1, i.e., after two periods on our time line. Thus, we need to compute* B_2. *Before doing that, we need to consider the per-period discount rate* r^*. *The bank rate* $r = 0.04$ *is the annual rate, we need to find the equivalent per-period rate; that is, we need to find* $APR/m = pp = r^*$ *that is equivalent to an* $EAR = 4\%$. *We proceed as follows:*

$$\begin{aligned} EAR &= (1+r^*)^m - 1 \\ &\Longrightarrow 0.04 = (1+r^*)^2 - 1 \\ &\Longrightarrow 1.04 = (1+r^*)^2 \\ &\Longrightarrow \sqrt{1.04} = 1 + r^* \\ &\Longrightarrow \sqrt{1.04} - 1 = r^* \\ &\Longrightarrow r^* = 0.0198 \ (or 1.98\%). \end{aligned}$$

Check: Notice that 1.98% is the semi-annual equivalent to 4% bank rate annually. Finally, we compute the PV of the bond as usual:

$$\begin{aligned} B_2 &= \frac{40}{(1+0.0198)^1} + \frac{1040}{(1+0.0198)^2} \\ &= 1039.2 \end{aligned}$$

PROBLEM 17. *Consider a two-year bond with face value* $F = 1000$, *a coupon rate* $c = 10\%$, *and coupon payments paid quarterly. Suppose the bond is currently traded at a discount price* $B_0 = 800$. *Answer the following questions:*

(1) What is the current yield on this bond?
(2) What is the capital gain on this bond if the investor decided to buy the bond today and hold it till maturity?
(3) What is the rate of return on this bond if the investor decided to buy the bond today and hold it till maturity?
(4) What is the yield to maturity on this bond?

(5) If the investor is considering to buy the bond at the end of the first year and to hold it till maturity. How much would the investor pay for that bond if the per-period (or the quarterly) market interest rate is 5%?

SOLUTION 17. *We are given the following information:*

$F = \$1000$

$C = c \times F = 0.10 \times 1000 = 100$

$Frequency = m = 4$ (quarterly)

Per-period payment is $C/m = 100/4 = 25$

$Maturity = n = 2$ years $\implies$ we have 8 payments (8 periods)

$B_0 = 800$

(1) The current yield is the per-period payment divided by the buy price:

$$\text{current yield } = \frac{C/m}{B_0} = \frac{25}{800} = 3.125\%$$

(2) The capital gain is the difference between the sale and buy price divided by the buy price. The sale price here is the face value since that is what you will get if you hold the bond to maturity. Thus,

$$\text{capital gain } = \frac{\text{Sale - buy}}{\text{buy}} = \frac{1000 - 800}{800} = 25\%$$

(3) The rate of return is the sum of the current yield and capital gain

$$\begin{aligned} \text{rate of return } &= \text{ current yield } + \text{ capital gain} \\ &= 3.125\% + 25\% \\ &= 28.125\% \end{aligned}$$

(4) The yield to maturity is the rate that equates today's value to the present value of all future payments. Since, we already converted the problem to per-period, we can compute the per-period YTM, denoted by YTM^*, from the following equation

$$B_0 = \frac{C/m}{(1 + YTM^*)^1} + \cdots + \frac{C/m + F}{(1 + YTM^*)^8};$$

that is,

$$800 = \frac{25}{(1 + YTM^*)^1} + \cdots + \frac{1025}{(1 + YTM^*)^8}$$

Using the approximation method:

$$YTM^* \approx \frac{C/m + \frac{F-B_0}{n\times m}}{\frac{F+B_0}{2}} = \frac{25+\frac{1000-800}{8}}{\frac{1000+800}{2}} = 5.56\%$$

To find the annual equivalent, we need to substitute $YTM^* = pp = APR/m = 5.56\%$ in the EAR formula:

$$\begin{aligned} EAR &= \left(1+\frac{APR}{m}\right)^m - 1 \\ &= (1+0.0556)^4 - 1 \\ &= 24.16\% \end{aligned}$$

(5) Notice here that the interest rate is already given per-period. Therefore, no transformation is needed. We can immediately proceed to find the present value of the bond as follows:

$$\begin{aligned} B_4 &= \frac{25}{(1+0.05)^1} + \frac{25}{(1+0.05)^2} + \frac{25}{(1+0.05)^3} + \frac{1025}{(1+0.05)^4} \\ &= 911.35 \end{aligned}$$

6. Stock Valuation

The main difference between stock valuation and bond valuation is the riskiness of future cash flows. In case of a bond, the future payments are fixed. The risk lies in the fluctuations of the market interest rate. Common stocks promise shareholders dividends for ever. Unlike coupon payments, these future dividends vary from one year to another. Another distinction is the fact that any firm has to pay its debts first. Thus, bond holders are always seen as **first claimants**. Common stock holders are entitled to the firm's residual cash flow; the remaining cash flow after all operating expenses, debt expenses, and taxes are paid. For that reason, stock holders are often known as **residual claimants.**

There are tow common forms of equity instruments: common shares and preferred shares. A **common share (CS)** is an equity instrument that represents ownership in a corporation and gives voting rights on major corporate decisions. Common shares are claims to share in net income and assets of a business. Common shares give periodic payments to their holders in the form of dividends. Common shares are considered long term securities since they don't have maturity dates. A **preferred share (PS)** is an equity instrument by which the owner is entitled a fixed dividend payment for ever. The preferred share dividends must be made before any dividends are paid to common shareholders. In this section, we focus on valuing CS and PS. We begin with CS.

Corporations issue shares of common stock to the public when they need to raise funds. To this end, they use the service of investment dealers such as RBC Capital Markets or BMO Capital Markets. Those dealers provide help in pricing and selling the common shares in the primary market. There are two types of primary market issues. An **initial public offering** (IPO) or **seasoned offering**. An IPO is a first offering of stock to the general public. In other words, if a privately owned company sells its stocks to the public for the first time, the issue is called an IPO. On the other hand, established firms that have already issued stocks to the public may decide to raise money by issuing additional shares. Sales of new shares are also primary market issues and are called seasoned offerings.

Secondary markets are markets where already-issued securities are traded among investors. These markets are called stock exchange markets. The major stock exchange market in Canada is the Toronto Stock Exchange, TSX, in which shares of large established corporations are listed. The TSX Venture Exchange, TSX-V, lists new, smaller corporations, including those that are specialized in manufacturing, technology, and natural resources. The CNQ is another Canadian stock exchange, in which listing of very small new ventures occur. The two major exchanges in the United States are the New York Stock Exchange (NYSE) and the NASDAQ market. The NYSE is an electronic auction market, whereas the NASDAQ exchange provides a screen on which investors can see the prices at which the dealers are prepared to trade. If the price is satisfactory, an investor can simply choose to strike a deal with the corresponding dealer. In an electronic exchange, each exchange maintain an electronic order book that shows offers to buy and sell with shares sold to the highest bidder. All Canadian exchanges are electronic exchanges.

Our objective in this section is to value a common stock. As usual, when we value any asset, we ought to look at the cash flows generated by the asset. The cash flows in our case are the expected dividends. These dividends, unfortunately, are not usually fixed, and thus, we ought to make assumptions regarding their expectations. We will discuss shortly different models of valuing stocks and the assumptions based on which these models are formed. For now, let's look at a very simple way of valuing common stocks.

Let P_0 denote the "**time zero**" price of a CS. This is simply today's price. Let $\widetilde{P}_1$ be "**time 1**" price. This is the **expected** price one year hence. Let the expected dividend per share one year from today be denoted by $\widetilde{D}_1$. The tilde on top of D and P_1 signifies that they are random variables. It is common in the finance literature to use the tilde symbol on top of any letter to denote a random variable. Notice that since P_0 is known to us, it doesn't include a tilde on top. In other words, it is treated as a constant and not a random variable. We are working under the assumption that dividends are paid once a year and the next dividend will be paid one year from now. Our objective is to compute the rate of return that investors expect from this CS over the next year. Recall that in the bond case, the rate of return (sum of current yield and capital gain), is certain. This is true since all future coupon payments are fixed payments, and thus, don't involve any uncertainty. In the case of CS, however, all future dividends are uncertain, i.e., random variables, and, thus, the rate of return on CS ought to be an **expected** return and not an actual (known) return. To emphasize this point, we will denote the actual rate of return by r, whereas its expectation will be denoted by $\widetilde{r}$. Therefore, like the case of bond, but under uncertainty, the expected rate of return on CS is defined as the sum of the expected dividend yield and the expected capital gain.

DEFINITION 37. *The expected common stock dividend yield is the expected dividend at the end of the year over the current price; that is,*

$$\textit{expected dividend yield} = \frac{\widetilde{D}_1}{P_0}.$$

DEFINITION 38. *The expected capital gain/loss at the end of the year is the expected change in stock price divided by the current common stock price; that is,*

$$\textit{expected capital gain} = \frac{\widetilde{P}_1 - P_0}{P_0}.$$

DEFINITION 39. *The expected rate of return on CS, denoted by $\widetilde{r}$, is the sum of the expected dividend yield and the expected capital gain; that is,*

$$\widetilde{r} = \frac{\widetilde{D}_1}{P_0} + \frac{\widetilde{P}_1 - P_0}{P_0}. \tag{5.18}$$

Isolating P_0 from (5.18) yields an expression of the current value of the CS in terms of its future one-period expected dividend and expected price as

$$P_0 = \frac{\widetilde{D}_1 + \widetilde{P}_1}{1 + \widetilde{r}}. \tag{5.19}$$

Notice that equation (5.19) is just the present value of the cash flows pertaining to the stock discounted at the expected rate of return $\widetilde{r}$. The variables in the right-hand-side of equation (5.19) are the investors' forecasts/predictions regarding return, dividend and price. In practice, investors make return predictions pertaining to a particular stock by looking at the rates of return of other stocks in the same risk class. The idea here is that, at any point in time, securities in the same risk, i.e., of the same risk, are priced to offer the same expected return. This is a fundamental characteristic of well functioning markets. For instance, if you are interested in assessing the expected rate of return of TD bank security, you can use the expected return of another security in the same risk class such as RBC for example. Both TD and RBC belong to the same industry, and thus, are expected to have the same risk. Here is an example that illustrates how formula (5.19) is used in practice to value a particular stock.

EXAMPLE 37. *AAA corporation announced that next year's dividend is expected to be \$10 per share. As an investor, you have the following prediction data regarding the corporation stock price with different probabilities at different states of the economy*

State i of the Economy	*Price:P_i*	*Probability:α_i*
Steady	*100*	0.5
Expansion	*105*	0.2
Recession	*92*	0.3

Thus, using the rule of expectation rule (see equation (2.3) in Chapter 2), we can compute the expected price next year as follows:

$$\begin{aligned} & EP \overset{or}{\equiv} \widetilde{P}_1 \overset{defn}{=} \sum P_i \alpha_i \\ = \;& 100 \times 0.5 + 105 \times 0.2 + 92 \times 0.3 \\ = \;& 98.6. \end{aligned}$$

If equally risky stocks, i.e., stocks pertaining to corporations in the mining industry, offer expected rate of return of 10%; that is, $\widetilde{r} = 0.1$, then

$$\begin{aligned} P_0 &= \frac{\widetilde{D}_1 + \widetilde{P}_1}{1 + \widetilde{r}} \\ &= \frac{10 + 98.6}{1 + 0.1} \\ &= 98.727. \end{aligned}$$

The valuation formula in (5.19), which requires tomorrow's stock price and the expected return, is not usually helpful since future prices are not easy to predict. One ought to make probabilistic judgements about future prices in different states of the economy. These figures are not usually accurate and are usually subjective. A better and more practical approach to valuation is one that expresses a stock's value as the present value of all forecasted future dividends paid by the corporation without referring to the future stock price. This method is known as the **dividend discount model**, which we now turn to.

7. The Dividend Discount Model

The dividend discount model is a model that computes the value of a stock today based on discounting all future cash flows, i.e., all future expected dividends, using an expected rate of return that is consistent with the company's risk class. One issue with forecasting future dividends is that, unlike bonds, stocks don't mature; rather they provide dividends for ever. Thus, according to this model, the present value of a stock is computed by the following discounting formula:

$$PV(stock) \overset{or}{=} P_0 = \frac{\widetilde{D}_1}{1+\widetilde{r}} + \frac{\widetilde{D}_2}{(1+\widetilde{r})^2} + \frac{\widetilde{D}_3}{(1+\widetilde{r})^3} + \cdots, \tag{5.20}$$

where $\widetilde{D}_i$ is the expected dividend at the end of year i. Notice that equation (5.20) goes for ever. In practice, however, we simplify the above formula by making reasonable assumptions regarding the time horizon and the growth of future dividends. In particular, we discuss three variants of the dividend discount model: the **no growth model**, the **constant growth model**, and the **variable growth model**. In the following three sub-sections, we discuss each model, its assumptions, and derive its valuation formula.

7.1. The No Growth Model. The no growth model assumes that future dividends are not growing. In other words, if the last dividend paid, denoted by D_0, is \$10 dollars say, then all future dividends ought to be equal to \$10. Although this assumption might not be realistic in practice, it simplifies the discounted cash formula in equation (5.20) tremendously. In fact, if all dividends are constant, then equation (5.20) is just a perpetuity:

$$PV(stock) \overset{or}{=} P_0 = \frac{D_0}{1+\widetilde{r}} + \frac{D_0}{(1+\widetilde{r})^2} + \frac{D_0}{(1+\widetilde{r})^3} + \cdots. \tag{5.21}$$

Using the present value formula of a perpetuity that we discussed in equation (4.10) in Chapter 4, it is easy to see that the present value of a stock with constant dividends is simply

$$PV(stock) \overset{or}{=} P_0 = \frac{D_0}{\widetilde{r}}. \tag{5.22}$$

The valuation formula in equation (5.22) states that the present value of a stock today depends on the last dividend paid and the expected rate of return, i.e., the rate of return offered by other equally risky stocks.

EXAMPLE 38. *Consider the common stock of AA corporation. Suppose that the current dividend is \$8 per share. If the rate of return on another equally risky stock in the same class is* 10% *and if you assume that dividends will remain constant for ever, then the present value of AA corporation is simply*

$$PV(stock) \overset{or}{=} P_0 = \frac{D_0}{\widetilde{r}} = \frac{8}{0.1} = \$80.$$

The assumption that dividends are constant forever is not sensible in most applications. A better assumption, perhaps, is to predict some sort of dividend growth. The next model allows dividends to grow at a constant growth rate over time.

7.2. The Constant Growth Model. The Constant growth model, which was proposed by Gordon (1962), is based on two assumptions: (1) dividends pertaining to the stock under consideration are expected to grow at a steady rate, g, for ever. This assumption is sensible for well established mature corporations. (2) the growth rate is less than the expected rate of return; that is, $g < \widetilde{r}$. This assumption ensures that the value of the stock today is not negative. The reason that $\widetilde{r} > g$ will become clear shortly. Let D_0 denote the current dividend. This is considered the last known dividend ③ paid on the stock under consideration. This value is presumed to be known. All future dividends are expected to be grow at a rate g; that is,

$$\widetilde{D}_1 = D_0(1+g) \tag{5.23}$$

$$\widetilde{D}_2 = \widetilde{D}_1(1+g), \tag{5.24}$$

$$\widetilde{D}_3 = \widetilde{D}_2(1+g), \tag{5.25}$$

$$\widetilde{D}_4 = \widetilde{D}_3(1+g),$$

and so on. Notice that, by recursive substitution, we can express all future dividends in term of the known current dividend D_0. To see this, substitute (5.23) in (5.24) to obtain

$$\begin{aligned} \widetilde{D}_2 &= \underbrace{\widetilde{D}_1}_{=D_0(1+g)} (1+g) \\ &= D_0(1+g)(1+g) \\ &= D_0(1+g)^2. \end{aligned} \tag{5.26}$$

Similarly, we can express $\widetilde{D}_3$ in term of D_0 by substituting (5.26) in (5.25). This gives

$$\begin{aligned} \widetilde{D}_3 &= \underbrace{\widetilde{D}_2}_{D_0(1+g)^2} (1+g) \\ &= D_0(1+g)^2(1+g) \\ &= D_0(1+g)^3. \end{aligned}$$

So, in general, at any time t,

$$\widetilde{D}_t = D_0(1+g)^t. \tag{5.27}$$

The stock valuation formula of the constant growth model is, therefore,

$$PV(stock) \overset{or}{=} P_0 = \frac{\widetilde{D}_1}{1+\widetilde{r}} + \frac{\widetilde{D}_2}{(1+\widetilde{r})^2} + \cdots + \frac{\widetilde{D}_t}{(1+\widetilde{r})^t} + \cdots,$$

or equivalently,

$$PV(stock) \overset{or}{=} P_0 = \frac{D_0(1+g)}{1+\widetilde{r}} + \frac{D_0(1+g)^2}{(1+\widetilde{r})^2} + \cdots + \frac{D_0(1+g)^t}{(1+\widetilde{r})^t} + \cdots. \tag{5.28}$$

Take D_0 as a common factor in (5.28). This gives

$$\begin{aligned} PV(stock) \overset{or}{=} P_0 &= D_0\left[\frac{(1+g)}{1+\widetilde{r}} + \frac{(1+g)^2}{(1+\widetilde{r})^2} + \cdots\right] \\ &= D_0\left[\left(\frac{1+g}{1+\widetilde{r}}\right)^1 + \left(\frac{1+g}{1+\widetilde{r}}\right)^2 + \cdots\right] \\ &= D_0 \sum_{j=1}^{\infty}\left(\frac{1+g}{1+\widetilde{r}}\right)^j \qquad (5.29) \end{aligned}$$

If $\widetilde{r} > g$, we can claim that the sum on the right-hand-side of equation (5.29) is simply $\frac{1+g}{\widetilde{r}-g}$. To see that

$$\sum_{j=1}^{\infty}\left(\frac{1+g}{1+\widetilde{r}}\right)^j = \frac{1+g}{\widetilde{r}-g}$$

multiply both sides by $1 - \frac{1+g}{1+\widetilde{r}}$. This gives

$$\left(1 - \frac{1+g}{1+\widetilde{r}}\right)\sum_{j=1}^{\infty}\left(\frac{1+g}{1+\widetilde{r}}\right)^j = \left(1 - \frac{1+g}{1+\widetilde{r}}\right)\frac{1+g}{\widetilde{r}-g},$$

or equivalently,

$$\sum_{j=1}^{\infty}\left(\frac{1+g}{1+\widetilde{r}}\right)^j - \sum_{j=1}^{\infty}\left(\frac{1+g}{1+\widetilde{r}}\right)^{j+1} = \left(\frac{1+\widetilde{r}}{1+\widetilde{r}} - \frac{1+g}{1+\widetilde{r}}\right)\frac{1+g}{\widetilde{r}-g},$$

or simply

$$\frac{1+g}{1+\widetilde{r}} = \left(\frac{\widetilde{r}-g}{1+\widetilde{r}}\right)\frac{1+g}{\widetilde{r}-g};$$

that is,

$$\frac{1+g}{1+\widetilde{r}} = \frac{1+g}{1+\widetilde{r}}.$$

Thus, in the constant growth model,

$$PV(stock) \overset{or}{=} P_0 = D_0\left(\frac{1+g}{\widetilde{r}-g}\right). \qquad (5.30)$$

Equation (5.30) allows us to evaluate a stock by knowing three pieces of information: (1) the last dividend paid, which is known to the investor. (2) The growth rate of the dividend g. This is assumed to be constant, which is a reasonable assumption to make for stable, well established, corporations. (3) The discount rate $\widetilde{r}$. This rate can also be estimated by looking at other stocks in the same risk class. Equation (5.30) states that today's stock price is higher if the last dividend paid is larger or if the growth rate of dividends is larger. These implications are sensible; if the dividend is larger or growing at a faster rate, the stock price ought to be higher. The equation also states that the current stock price is higher if the discount rate is $\widetilde{r}$

is smaller. Is this implication sensible? Well, yes it is. Recall that, in the case of bond valuation, r was the yield to maturity, which was negatively related to the bond price. In our case, $\widetilde{r}$ is the required (or expected) rate of return on equity. It could be though of as the yield to maturity on a portfolio of discount bonds with the same risk characteristics as the stock. And since, lower yield implies higher price, the implication of this relation is also sensible.

① Notice also that the valuation formula of the constant growth model doesn't depend on the time horizon. Investors who have different time horizons should arrive at the same valuation as long as they make the same assumptions regarding the growth rate of the stock and the discount rate.

It is worth noting that the discount rate $\widetilde{r}$ in equation (5.30) has two meanings. From the investor's point of view, it is the **required** or **expected rate of return** on equity. It is the expected rate of return offered by other equally risky stocks. From the firm's perspective, it is the **cost** of equity, which we will denote by k_s. Thus, the formula in equation (5.30) can be used by an investor to calculate the present value of a stock, or by a corporation to calculate the cost equity. In the latter case, $\widetilde{r} \overset{or}{=} k_s$ is taken to be the cost of equity. Solving for k_s yields

$$k_s = \frac{D_0(1+g)}{P_0} + g. \tag{5.31}$$

Notice, that if $g = 0$, i.e., if dividends are assumed to be constant, then

$$k_s = \frac{D_0}{P_0}. \tag{5.32}$$

REMARK 13. *It is useful to remember that in the constant growth formula, the numerator is just* $\widetilde{D}_1$*; that is,*

$$P_0 = \frac{D_0\,(1+g)}{\widetilde{r}-g} = \frac{\widetilde{D}_1}{\widetilde{r}-g}.$$

Thus, to compute the price of a stock today, we only need the dividend one-period ahead. Since, the time horizon is irrelevant, i.e., we only need the one-period ahead dividend, we can use the constant growth formula at any point in time; namely,

$$P_t = \frac{D_{t+1}}{\widetilde{r}-g}. \tag{5.33}$$

This remark is particularly useful in the discussion on the variable growth model which we will introduce in the following section.

Below are some examples.

EXAMPLE 39. *Consider the common stock of AAA corporation. Suppose that the last dividend paid was \$3 per share. The firm is growing at a steady* *rate and investors expect that the dividends to grow at a rate of 8% per year. Suppose that the expected rate of return offered by another equally risky stock is 12%. Then, the present value of this stock using Gordon's constant growth model is*

$$PV(stock) \overset{or}{=} P_0 = D_0 \left(\frac{1+g}{\widetilde{r}-g} \right) = 3 \times \left(\frac{1+0.08}{0.12-0.08} \right) = \$81.$$

EXAMPLE 40. *Consider again the common stock of AAA corporation. Suppose that AAA wishes to compute its cost of equity. If the corporation makes the same growth assumption about its dividends and if the stock is currently traded at a price of \$81. Then, it is easy to see from equation (5.31) that the cost of equity* $ks = 12\%$.

PROBLEM 18. *Suppose that the last dividend paid on IBM share is \$100. The dividends are expected to grow at a constant growth rate of 4% per year. Your financial analyst managed to collect the following data on Intel share (a similar stock at the same risk class):*

Scenario	***Probability***	***Rate of return on Intel***
Bust	0.5	−14%
Boom	0.5	38%

Determine the fundamental market price of IBM, i.e., the current value of IBM common shares.

SOLUTION 18. *In this problem, we are given data to compute the expected rate of return,* $\widetilde{r}$, *or the discount rate, on a similar security in the same risk class of IBM. Using the rule of expectation, we have*

$$\widetilde{r} = Er = 0.5(-14) + 0.5(38) = 12\%.$$

It follows then that

$$\begin{aligned} P_0 &= \frac{\widetilde{D}_1}{\widetilde{r}-g} \\ &= \frac{D_0(1+g)}{\widetilde{r}-g} \\ &= \frac{100(1+0.04)}{0.12-0.04} = 1300. \end{aligned}$$

7.3. Variable Growth Model. The discounting formula of the constant growth model in equation (5.30) is better than the simple expected return formula in equation (5.19) because it doesn't include the forecasted price of the stock. It is also preferred over the no growth formula in equation (5.22) since the constant growth assumption is more sensible. However, in practice, many companies grow at a rapid or irregular rates for several years before finally settling down. The variable growth model takes into account this situation. In this model, the firms' dividends are expected to grow at a rapid growth rate, g_1, over a few years, then at constant growth rate, g, onward for ever. If we let the time horizon over which the dividends are growing at a rapid growth be t, then the preset value of the firm's stock could be calculated as follows

$$\begin{aligned} PV(stock) \overset{or}{=} P_0 &= \frac{D_0(1+g_1)}{1+\widetilde{r}} + \frac{D_0(1+g_1)^2}{(1+\widetilde{r})^2} + \cdots + \frac{D_0(1+g_1)^t}{(1+\widetilde{r})^t} \\ & \quad + PV\left[\underbrace{\frac{D_{t+1}}{\widetilde{r}-g}}_{=P_t}\right], \end{aligned} \tag{5.34}$$

where the last term is the present value of the price at t, which could be obtained using Gordon's constant growth formula at time t discounted for t periods (see equation (5.33) in Remark 15). Notice that

$$D_{t+1} = D_t(1+g).$$

For instance, if dividends are expected to grow at a rapid rate of g_1 over $t = 3$ years and at a rate g thereafter, then

$$\begin{aligned} P_0 &= \frac{\widetilde{D}_1}{1+\widetilde{r}} + \frac{\widetilde{D}_2}{(1+\widetilde{r})^2} + \frac{\widetilde{D}_3}{(1+\widetilde{r})^3} + PV\left[P_3\right] \\ &= \frac{D_0(1+g_1)}{1+\widetilde{r}} + \frac{D_0(1+g_1)^2}{(1+\widetilde{r})^2} + \frac{D_0(1+g_1)^3}{(1+\widetilde{r})^3} + PV\left[\widetilde{D}_3\left(\frac{1+g}{\widetilde{r}-g}\right)\right] \\ &= \frac{D_0(1+g_1)}{1+\widetilde{r}} + \frac{D_0(1+g_1)^2}{(1+\widetilde{r})^2} + \frac{D_0(1+g_1)^3}{(1+\widetilde{r})^3} + PV\left[\frac{\widetilde{D}_4}{\widetilde{r}-g}\right] \end{aligned}$$

EXAMPLE 41. *A company is expected to pay a dividend of \$10 at the end of this year. The dividends are expected to grow at a rapid growth rate of 10% over the next three years, then at a constant growth rate of 4% per year thereafter. Determine the market price, i.e., current value of this company's common shares if the required/expected rate of return is 12%. First, we compute the dividends over the rapid growth period of three years as follows:*

$$\begin{aligned} \widetilde{D}_1 &= D_0(1+g_1) = 10(1+0.1) = 11 \\ \widetilde{D}_2 &= D_0(1+g_1)^2 = 10(1+0.1)^2 = 12.1 \\ \widetilde{D}_3 &= D_0(1+g_1)^3 = 10(1+0.1)^3 = 13.31. \end{aligned}$$

Next, we compute the first dividend paid at the expected constant growth rate $g = 4\%$ *as follows*

$$\widetilde{D}_4 = D_3(1+g) = 13.31(1+0.04) = 13.8424.$$

Notice that to compute P_3, *we only need* $\widetilde{D}_4$. *Thus, all other dividends are irrelevant. Using Gordon's constant growth formula, we can find the price* *of the stock at time $t = 3$ as follows*

$$\widetilde{P}_3 = \frac{\widetilde{D}_4}{\widetilde{r} - g} = \frac{13.8424}{0.12 - 0.04} = 173.03.$$

The total present value is, therefore, the sum of the present values of the first three dividends plus the present value of P_3: that is,

$$\begin{aligned} P_0 &= \frac{\widetilde{D}_1}{1+\widetilde{r}} + \frac{\widetilde{D}_2}{(1+\widetilde{r})^2} + \frac{\widetilde{D}_3}{(1+\widetilde{r})^3} + PV\left[\frac{\widetilde{D}_4}{\widetilde{r} - g}\right] \\ &= \frac{11}{(1+0.12)} + \frac{12.1}{(1+0.12)^2} + \frac{13.31}{(1+0.12)^3} + \frac{173.03}{(1+0.12)^3} \\ &= 152.1 \end{aligned}$$

PROBLEM 19. *Suppose that the last dividend paid on IBM share is \$100. The dividends are expected to grow at a rapid growth rate of 10% over the next three years, then at a constant growth rate of 4% per year thereafter. Your financial analyst managed to collect the following data on Intel share (a similar stock at the same risk class):*

Scenario	***Probability***	***Rate of return on Intel***
Bust	0.5	−14%
Boom	0.5	38%

Determine the fundamental market price of IBM, i.e., the current value of IBM common shares.

SOLUTION 19. *First, we compute the dividends over the rapid growth period of three years as follows:*

$$\begin{aligned} \widetilde{D}_1 &= D_0(1+g_1) = 100(1+0.1) = 110 \\ \widetilde{D}_2 &= D_0(1+g_1)^2 = 100(1+0.1)^2 = 121 \\ \widetilde{D}_3 &= D_0(1+g_1)^3 = 100(1+0.1)^3 = 133.1. \end{aligned}$$

Next, we compute the first dividend paid at the expected constant growth rate $g = 4\%$ as follows

$$\widetilde{D}_4 = D_3(1+g) = 133.1(1+0.04) = 138.424.$$

Notice that to compute P_3, we only need $\widetilde{D}_4$. Thus, all other dividends are irrelevant. Using Gordon's constant growth formula, we can find the price of the stock at time $t = 3$ as follows

$$P_3 = \frac{\widetilde{D}_4}{\widetilde{r} - g} = \frac{138.424}{0.12 - 0.04} = 1730.3.$$

Now, we compute the expected rate of return or the discount rate $\widetilde{r}$ using the data given in the problem and the rule of expectation as follows:

$$\widetilde{r} = Er = 0.5(-14) + 0.5(38) = 12\%$$

The total present value is, therefore, the sum of the present values of the first three dividends plus the present value of P_3; that is,

$$\begin{aligned} P_0 &= \frac{\widetilde{D}_1}{1+\widetilde{r}} + \frac{\widetilde{D}_2}{(1+\widetilde{r})^2} + \frac{\widetilde{D}_3}{(1+\widetilde{r})^3} + PV\left[\frac{\widetilde{D}_4}{\widetilde{r}-g}\right] \\ &= \frac{110}{(1+0.12)} + \frac{121}{(1+0.12)^2} + \frac{133.1}{(1+0.12)^3} + \frac{1730.3}{(1+0.12)^3} \\ &= \$1521 \end{aligned}$$

8. Valuing Preferred Stocks

Preferred shares offer fixed dividends for ever. The cash stream of a preferred share is, therefore, a perpetuity with a present value calculated as a ratio of the fixed payment, i.e., the fixed dividend, over the discount rate. The discount rate is, as usual, the required rate of return (or the expected return on another equally risky instrument) from the investor's point of view, or the cost of preferred equity, denoted by k_p, from the corporation's point view. Thus, the preferred stock valuation formula is

$$P_0 = \frac{D}{\widetilde{r}} = \frac{D}{k_p} \tag{5.35}$$

and

$$k_p = \frac{D}{P_0}. \tag{5.36}$$

Valuing stocks is always harder in practice than in principle. The trouble lies in the assumptions imposed on the dividend discount model. In particular, forecasting cash flows and finding an appropriate discount rate require skill and judgement. In practice, fund and portfolio managers rely on financial analysts in gathering information regarding stocks, bonds, and other assets. Every fund has an army of analysts, who conduct thorough analysis on financial instruments. Fund managers use this information in constructing their portfolios. A process known as **asset allocation and portfolio construction**.

9. Some Practice Problems on Stock Valuation

PROBLEM 20. *ABC Corporation will pay a year-end dividend of \$2.50 per share. Investors expect the dividend to grow at a rate of 4% indefinitely. (a) If the stock currently sells for \$25 per share, what is the expected rate* *of return on the stock? (b) Suppose you are making your own expectation about the rate of return on this stock using the following data:*

States	*return*	*Probability*
Expansion	20%	0.5
Steady	13%	0.5

What is the stock price?

SOLUTION 20. *(a) This is a constant growth model. The price of the stock,* P_0*, from this model is the present value of all future dividends, which is given by the equation*

$$P_0 = \frac{\widetilde{D}_1}{\widetilde{r} - g},$$

where $P_0 = 25$, $\widetilde{D}_1 = D_0(1+g) = 2.5(1+0.04) = 2.6$, *and* $P_0 = 25$. *Here, we are trying to solve for the expected rate of return,* $\widetilde{r}$. *Thus,*

$$\widetilde{r} = \frac{\widetilde{D}_1}{P_0} + g = \frac{2.6}{25} + 0.04 = 14.4\%.$$

(b) Here the expected return is $Er = 20(0.5) + 13(0.5) = 16.5\%$, *which is taken to be* $\widetilde{r}$ *in the constant growth formula. We need to compute* P_0. *Using the same formula, we get*

$$P_0 = \frac{\widetilde{D}_1}{\widetilde{r} - g} = \frac{2.6}{16.5 - 4} = \$20.8.$$

PROBLEM 21. *Sweet Potatoes Corp. paid \$1 per share dividend yesterday. You expect the dividend to grow steadily at 4 percent per year. (a) What is the expected dividend in each of the next three years? (b) If the discount rate for the stock is 12 percent, at what price will the stock sell? (c) What is the expected price three years from now? (d) If you buy the stock and plan to hold if for three years, i.e., you are selling it at the end of year three, what payments will you receive? What is the present value of those payments? Compare your answer to part (b).*

SOLUTION 21. *(a) Here we have* $D_0 = 1$ *and* $g = 0.04$. *Thus,*

$$\begin{aligned} D_1 &= D_0(1+g) = 1(1+0.04) = 1.04 \\ D_2 &= D_0(1+g)^2 = 1.0816 \\ D_3 &= D_0(1+g)^2 = 1.1249 \end{aligned}$$

(b) Given that $r = 0.12$, *we can compute* P_0 *as follows:*

$$P_0 = \frac{\widetilde{D}_1}{\widetilde{r} - g} = \frac{1.04}{0.12 - 0.04} = \$13$$

(c) Here, we need to compute P_3. *First, we need to find* D_4 *as follows:*

$$D_4 = D_0(1+g)^4 \overset{or}{=} D_3(1+g) = 1.1249(1+0.04) = 1.1699$$

Thus,

$$P_3 = \frac{\widetilde{D}_4}{\widetilde{r} - g} = \frac{1.1699}{0.12 - 0.04} = \$14.623.$$

(d) Holding the stock only for three periods. You will receive the first three dividends and the sale price at end of year 3, which is $P_3 = 14.623$. *The present value of those payments is*

$$\begin{aligned} PV &= \frac{D_1}{(1+r)^1} + \frac{D_2}{(1+r)^2} + \frac{D_3 + P_3}{(1+r)^3} \\ &= \frac{1.04}{(1+0.12)^1} + \frac{1.0816}{(1+0.12)^2} + \frac{1.1249 + 14.623}{(1+0.12)^3} \\ &= \$13 \end{aligned}$$

This is less than the answer in part (c) simply because you are not holding the stock for ever.

PROBLEM 22. *You expect a share of stock to pay dividends of \$1, \$1.25, and \$1.50 in each of the next three years. You believe that the stock will sell for \$20 at the end of the third year. (a) What is the stock price if the discount rate for the stock is 10 percent? (b) What is the dividend yield?*

SOLUTION 22. *(a) The stock price today,* P_0, *is the present value of all future payments; that is,*

$$\begin{aligned} P_0 &= \frac{D_1}{(1+r)^1} + \frac{D_2}{(1+r)^2} + \frac{D_3 + P_3}{(1+r)^3} \\ &= \frac{1}{(1+0.1)^1} + \frac{1.25}{(1+0.1)^2} + \frac{1.5 + 20}{(1+0.1)^3} \\ &= \$18.1. \end{aligned}$$

(b) The dividend yield is, by definition,

$$\textit{dividend yield} = \frac{D_1}{P_0} = \frac{1}{18.1} = 5.52\%.$$

PROBLEM 23. *Web Cites Research's stock is currently traded at \$23.3. Management plan to plow back 30 percent of all earnings into the firm. Earning this year will be \$2 per share, and investors expect a 12 percent rate of return on the stock. (a) What is the sustainable growth rate? (b) What is the P/E ratio? (c) What would be the share price and the P/E ratio if the firm paid out all earnings as dividends?*

SOLUTION 23. *(a) Here we know that* $P_0 = 23.3$. *For every \$2 earnings, 30% will be retained, i.e., 0.60 will be retained and 1.4 will be paid as dividends. Thus,* $\widetilde{D}_1 = 1.4$. *Dividends are growing at a constant/sustainable growth rate of* g, *which is the rate that we are trying to compute. Using the constant growth formula, we have*

$$P_0 = \frac{\widetilde{D}_1}{\widetilde{r} - g} \implies 23.3 = \frac{1.4}{0.12 - g} \implies g = 6\%.$$

((b)

$$P/E = \frac{Price\ per\ share}{Earning\ per\ share} = \frac{23.3}{2} = 11.65.$$

(c) In this case $\widetilde{D}_1 = 2$, $\widetilde{r} = 0.12$, *and* $g = 0.06$. *Thus,*

$$P_0 = \frac{\widetilde{D}_1}{\widetilde{r} - g} = \frac{2}{0.12 - 0.06} = 33.3$$

and

$$P/E = \frac{Price\ per\ share}{Earning\ per\ share} = \frac{33.3}{2} = 16.67.$$

PROBLEM 24. *Trend Inc. has been growing at 6 percent rate per year and it is expected to continue to do so indefinitely. The next dividend is expected to be $5 per share. (a) If the market expects a 10 percent rate of return on Trend Inc., at what price must it be selling, i.e., what is the fundamental price? (b) If the stock is currently selling for $100. Provided that the analysis in part (a) is correct, is there any arbitrage opportunity that you can exploit? Explain and relate your answer to the notion of market efficiency (see Chapter 3). (c) Suppose you decided to exploit this arbitrage opportunity by constructing a portfolio that consists of this stock and a risk-free asset that pays a rate of return of 2 percent. What is the expected rate of return on your portfolio if you decided to allocated equal weights to both assets?*

SOLUTION 24. *Here, we have* $g = 0.06$, $\widetilde{D}_1 = 5$, *and* $\widetilde{r} = 0.10$. *Therefore,*

$$P_0 = \frac{\widetilde{D}_1}{\widetilde{r} - g} = \frac{5}{0.10 - 0.06} = \$125.$$

This is the fundamental or intrinsic value of the stock. (b) If the current price is $100. Then the stock is under-priced. One can buy and hold the stock in a portfolio to benefit from the expected gains in the future when the price reaches its fundamental value. If our constant growth assumption is correct, then the fundamental price of $125 is the efficient price. It is the price that fully reflect all available information according to the EMH. The traded price of $100 is inefficient since it is below the fundamental price. Inefficiency creates profit opportunities. In this case, one can make profit by adopting a simple buy-and-hold strategy. (c) The expected rate of return on the portfolio is computed using the expectation formula as follows:

$$\begin{aligned} Er_p &= w_1 Er_1 + w_2 Er_2 \\ &= 0.5(10\%) + 0.5(2\%) \\ &= 6\% \end{aligned}$$

10. Practice Problems

(1) Define what it means by the yield to maturity on a bond.

(2) Explain the relation between market rate and bond prices. No calculations are needed.

(3) ABC bond has 10 years until maturity, face value of \$1000, and a coupon rate of 8 percent payable annually. The bond sells today (end of year 9) for \$1050. Answer the following questions.
   (a) What is the current yield on the bond?
   (b) Define what it means by the yield to maturity. What is the yield to maturity?

(4) Consider a two-year bond with face value $F = 1000$, a coupon rate $c = 10\%$, and coupon payments paid semiannually. Suppose the bond is currently traded at a discount price $B_0 = 750$. Answer the following questions:
   (a) What is the current yield on this bond?
   (b) What is the capital gain on this bond if the investor decided to buy the bond today and hold it till maturity?
   (c) What is the rate of return on this bond if the investor decided to buy the bond today and hold it till maturity?
   (d) What is the yield to maturity on this bond?
   (e) If the investor is considering to buy the bond at the end of the first year and to hold it till maturity. How much would the investor pay for that bond if the per-period (or the semi-annual) market interest rate is 5%?

(5) Consider a one-year bond with face value $F = 1000$, a coupon rate $c = 10\%$, and coupon payments paid quarterly. If the bond is currently traded at a discount at a price $B_0 = 800$. What is the rate of return on this bond if you bought it today at 800 and you are planning to hold it till maturity?

(6) Consider a one-year bond with face value $F = 1000$, a coupon rate $c = 10\%$, and coupon payments paid quarterly. If the bond is currently traded at a discount at a price $B_0 = 800$. What is the yield to maturity if you bought it today at 800 and you are planning to hold it till maturity?

(7) Consider a one-year bond with face value $F = 1000$, a coupon rate $c = 10\%$, and coupon payments paid quarterly. Suppose that the market interest rate is 5%. You are planning to buy the bond today. What is the price of this bond today, i.e., the present value of this bond?

(8) Consider a three-year bond with face value $F = 1000$ and a coupon rate $c = 10\%$ paid semi-annually. Suppose the bond is traded at a price $B_0 = 1025$. Answer the following questions:
   (a) What is the current yield on this bond?
   (b) What is the capital gain on this bond if the investor decided to hold the bond till maturity?
   (c) What is the rate of return on this bond?

  (d) Define what it means by yield to maturity and explain why it is better than the conventional rate of return.
  (e) Compute the six-month (per-period) yield to maturity on this bond. You can use the approximation formula or the trial and error method in your computation.
  (f) Compute the annual yield to maturity on this bond taking into account compound interest.
  (g) Suppose you bought that bond from this investor at the end of year 2. How much would you pay for that bond if the market interest rate is 5%?

(9) A company is expected to pay a dividend of \$5 at the end of this year. The dividends are expected to grow at a rapid growth rate of 8% over the next four years, then at a constant growth rate of 3% per year thereafter. Determine the market price, i.e., the current value of this company's common shares if the required rate of return is 12%.

(10) Suppose that the last dividend paid on IBM share is \$100. The dividends are expected to grow at a rapid growth rate of 10% over the next three years, then at a constant growth rate of 4% per year thereafter. Your financial analyst managed to collect the following annual data on Intel share (a similar stock at the same risk class):

Year	Return on Intel stock (2%)
1	12
2	14
3	12
4	10

Determine the fundamental market price of IBM, i.e., the current value of IBM common shares.

(11) Suppose that the last dividend paid on AAA share is \$100. The dividends are expected to grow at a rapid growth rate of 10% over one year, then at a constant growth rate of 4% per year thereafter. If the expected rate of return on AAA stock is 12%, determine the present value, i.e., the market price of that stock.

(12) Consider the preferred stock of *BBB* corporation. Suppose that the current preferred dividend is \$8 per share. If the rate of return on another equally risky preferred stock in the same class is 10%, find the price of this preferred share.

(13) Suppose that the last dividend paid on IBM share is \$100. The dividends are expected to grow at a constant growth rate of g. Your financial analyst managed to collect the following data on Intel share (a similar stock at the same risk class):

Scenario	**Probability**	**Rate of return on Intel**
Bust	0.5	−14%
Boom	0.5	38%

If the fundamental market price of IBM is 5500, determine g.

(14) True/False and Explain:

  True (a) The fundamental price of a stock obtained from the constant growth model will vary according to the investor's time horizon (long or short-term)

  True (b) The fundamental price of a preferred share depends on the expected preferred share dividend in future periods.

  True (c) The fundamental price of a constant-growth stock depends on the analyst's expectations about all future dividends.

  (d) The fundamental price of a stock or a bond is calculated using a variant of the DCF model and it is considered the efficient market price.

CHAPTER 6

Capital Budgeting

As mentioned earlier in Chapter 5, **valuation** is a key concept in finance. It is used in **finance theory**, **portfolio management**, **corporate finance**, and **capital theory**. In finance theory, computing the intrinsic or fundamental value of an asset, which is basically its fundamental price, and assessing whether this value is above or below the market value is a simple test of the Efficient Markets Hypothesis, which is the fundamental hypothesis in any financial model of price formation. Valuation is also used in **portfolio management**; when the difference between the intrinsic value of an asset and its market is negative (positive), the asset is said to be undervalued (overvalued). The price of an undervalued asset is expected to increase and, therefore, it is recommended to be included in the portfolio. The opposite is true for overvalued assets. In **corporate finance theory**, computing the fundamental value of a corporation through using a variant of the basic DCF formula given in equation (5.2), and conducting **ratio analysis** are routine tasks used by analysts to assess the performance of the corporation among its peers in the industry. Such comparison can provide management with useful insights on how to increase the value of the corporation by restructuring the corporation's investment, financial, and dividend decisions. In such case the asset under valuation is the corporation itself. Alternatively, the corporation could be interested in valuing investment projects for the purpose of assessing their viability or simply for the purpose of ranking them. The valuation of an investment project is known as **capital budgeting**. In this chapter, we will focus our attention on capital budgeting. Like any asset valuation, we will learn shortly that capital budgeting is also based on the fundamental DCF formula in equation (5.2).

We begin the discussion on capital budgeting by exploring the discount rate used in the DCF formula. This rate as known as the weighted average cost of capital. It is the weighted average of the various costs pertaining to the various instruments used by the corporation to raise capital. It is assumed that the reader brings the background of an introductory course in accounting, where exposure to basic notions such as financial statements, sources and uses of funds, and financial ratios is expected. Here at the outset, we briefly review some of these notions.

A quick inspection of the right-hand-side of a consolidated balance sheet reveals that any firm can raise funds through borrowing from a third party. This third party could be a financial intermediary, e.g., a financial bank. In this case, the borrowing is in the form of a short-term or long-term loan. This type of financing is known as **debt financing** or **fixed-cost** financing since the cost of financing, which is the interest rate on the loan, is fixed. Another type of debt financing is raising funds by issuing a corporate bond. Again, financing through issuing bonds is considered fixed-cost financing. Alternatively, the firm could raise funds by issuing common shares (CS) or preferred shares (PS). Borrowing via issuing CS or PS is considered **equity financing**. The key difference between equity financing and debt financing lies in the finance cost. Bond holders are entitled to receive a **fixed** amount known as bond interest every specific period of time till maturity. On the other hand, CS holders are entitled to receive dividends according to the firm's dividend policy. Dividend payments are naturally variable, and are allocated from the firm's net income at the end of the fiscal year as per the firm's dividend policy. In general, the firm could finance its capital through four main sources of funds: (1) **long-term debt** (**LTD**), (2) **preferred stocks**, (3) **common stocks**, and (4) **retained earnings** (**RE**). They can all be seen from the right-hand-side of the firm's balance sheet. In what follows, we examine the cost of each source of financing on an after-tax basis. The ultimate objective is to deduce a formula for the firm's weighted average cost of capital, which is taken to be the discount rate in the DCF formula for capital budgeting.

1. The Weighted Average Cost of Capital

1.1. Cost of LTD. A long-term debt is either in the form of a bank loan or a corporate bond. The cost of a bank loan to the borrower, i.e., the cost to the corporation, is simply the interest rate charged by the bank. The cost of a corporate bond, k_b, is calculated from equation (5.16) or from the approximation in (5.17). In capital budgeting, we will use the **after tax** approximation formula to calculate the cost of long-term debt in the form of a corporate bond. More formally, we define the before tax cost of debt k_b and the after tax cost of debt, denoted by k_d, as follows.

DEFINITION 40. *The* ***before-tax cost of debt***, *k_b, is computed from the equation*

$$k_b = \frac{C + \frac{F - N_b}{n}}{\frac{N_b + F}{2}}, \tag{6.1}$$

where C is the annual interest payment (coupon payment) on the debt instrument, n is the number of periods to maturity, N_b is the net proceeds from the sale of debt, i.e., the proceeds from the sale of the debt instrument *(bond), B, minus any flotation costs, and F is the bond's face or par value. The* ***after-tax cost of debt****, denoted by k_d, is computed from the equation*

$$k_d = k_b(1 - \tau_c), \tag{6.2}$$

where τ_c is the corporate tax-rate.

Equation (6.1) is very similar to the approximation in equation (5.17). The only difference is that the bond price B is adjusted for flotation costs. These are the costs incurred by the firm when issuing a debt instrument. Aside from this minor adjustment, everything else in (6.1) is the same as before. One point to remember here is that any value in the computation of the weighted average cost of capital must be after tax. So, k_d is what should be included in the weighted average cost of capital and not k_b.

1.2. Cost of CS. As we discussed in Chapter 5, the cost of common stock, k_s, is simply the **required rate of return** (**RRR**), also known as the **expected rate of return**, by investors in the market place. It is the minimum rate of return that an investor requires to acquire a common stock. This rate of return should be greater than or equal to the investor's opportunity cost of funds. There are alternative formulas that are commonly used in computing the RRR on a stock; each is a result of a financial equilibrium model or theory. For instance, one way to compute k_s is to use the capital asset pricing model that we discussed in Chapter 3. In this case, we can compute k_s as follows:

$$k_s = rf + \beta\left(Er_M - rf\right).$$

If the CAPM is the model of choice to compute the cost of equity, it is important to note that the value of the cost of equity obtained from this model is calculated under the presumption that all the 5 axioms of CAPM hold.

An alternative way of computing k_s is one that is derived from Gordon's (1962) stock valuation model that we discussed in Chapter 5. Recall that, according to Gordon's model, the current value of a common stock today, denoted by P_0, is calculated by the equation

$$P_0 = \frac{D_1}{k_s - g},$$

where $D_1 = D_0(1+g)$ is the **expected** dividend per share, D_0 is the current or last dividend received, and g is the growth rate of dividends. Thus, knowing D_0, g, and P_0, from Gordon's model, one can easily compute k_s as $k_s = \frac{D_1}{P_0} + g$. In this chapter, we will consider this formula as a definition of the cost of common stock.

Definition 41. *The* ***cost of common stock****, k_s, is the required rate of return by investors in the market place. It can be calculated using Gordon's (1962) model as*

$$k_s = \frac{D_1}{P_0} + g, \tag{6.3}$$

where P_0 is the stock price, $D_1 = D_0(1+g)$, g is the growth rate of dividends, and D_0 is the last dividend paid.

Notice here that since common share dividends are paid out of the firm's after-tax cash flows, adjusting the cost of common stock, k_s, is not required. The same argument stands for preferred shares.

1.3. Cost of PS. Preferred shares promise a fixed dividend, D_p, for ever. The fundamental price of a preferred share is the present value of a perpetuity, and it is calculated, as we already discussed in Chapter 5 (see equation (5.36)), as follows:

$$P_p = \frac{D_p}{k_p},$$

where k_p is the cost of preferred equity. Thus, we can compute k_p from the previous equation as

$$k_p = \frac{D_p}{P_p}.$$

Finally, to account for any flotation costs pertaining to issuing the preferred shares, we use the net proceeds, N_p, which is the share price minus flotation costs, instead of P_p in the denominator of the previous formula.

Definition 42. *Let D_p be the dividend paid on preferred stocks, and N_p be the net proceeds from the sale of the preferred stock. Then the* ***cost of PS****, k_p, is calculated by the equation*

$$k_p = \frac{D_p}{N_p}. \tag{6.4}$$

Preferred dividends are paid out of the firm's after-tax cash flows. Therefore, adjusting the cost, k_p, for taxes is not required.

1.4. Cost of RE. RE is another source of funds for any corporation. From an accounting sense, RE increases equity in the same way as the sale of additional shares of common stock. Therefore, the cost of RE to the firm, denoted by k_{RE}, should be treated in the same way as the cost of an equivalent **new issue** of additional common stocks, denoted by k_N. If one entertains Gordon's model to compute the cost of new issue of common stocks, then

$$k_{RE} = k_N = \frac{D_1}{N_n} + g, \tag{6.5}$$

where N_n is the net proceeds from the new issue, which is computed as

$$N_n = P_0 - \text{ flotation costs } - \text{ under pricing costs},$$

and everything else is the same as before.

Definition 43. *The* ***cost of retained earning****, k_{RE}, is equivalent to the cost of new issue of common stocks, k_N, which is computed by the equation*

$$k_N = \frac{D_1}{N_n} + g. \tag{6.6}$$

Notice that, due to flotation costs, the cost of new common stocks is normally greater than the cost of any other long-term financing, i.e., $k_N > k_{RE}$ and $k_N > k_s$.

Now, since we know how to compute the cost of each source of funds separately, we can simply define the cost of capital to be the weighted average of these costs.

Definition 44. *The* ***weighted average cost of capital*** *(****WACC****) is a weighted average of all the previous four costs with the weights being the proportion of each source of fund in the firm's capital structure; that is,*

$$WACC = w_d k_d + w_p k_p + w_s k_s + w_{RE} k_{RE}, \tag{6.7}$$

where w_i is the proportion of source of capital i in the firm's capital structure.

The WACC is an important tool in making investment decisions in new projects. It can be thought of as the rate of return a firm must earn on its investment projects in order to maintain the market value of its stock. On the other hand, from the demand side, it can also be thought of as the RRR by the suppliers of capital to invest their funds in the firm. As a rule of thumb, projects with rates of return above the firm's cost of capital should increase the value of the firm and vice-versa. The previous rule obviously assumes that the business and financial risks are held fixed. Therefore, to simplify the analysis of isolating the basic structure of the cost of capital of the firm under consideration, we will set forth the following simplifying assumptions regarding risks and taxes:

1. Business risk, which is the risk to the firm of being unable to cover its operating costs, is assumed to be unchanged in the short-run.
2. Financial risk, which is the risk to the firm of being unable to cover its financial obligations, e.g., paying interest due on debt, is also assumed to be unchanged in the short-run.
3. The cost of capital is measured on an after-tax basis.

CRITERION 2. *Under the presumption that the previous three simplifying assumptions are satisfied, the shareholders' criterion to invest in new investment projects is such that*

$$RRR > WACC. \tag{6.8}$$

Capital budgeting is defined as the process of selecting and evaluating long-term investments in a way that is consistent with the firm's general goal, which is shareholders' wealth maximization. The acquisition of real assets is an example of capital budgeting. There are many motives behind firms' need to capital budgeting. The main motives are **expansion**, **replacement**, and **renewal**. Expanding the level of operations is a common motive for capital expenditures. The expenditures here take the form of acquiring new real assets. Capital expenditures are also made to replace old or worn out assets. The motive for capital expenditures, in this case, is simply replacement of old assets with new ones. An alternative to replacement is renewal, e.g., rebuilding, or renewing, an existing asset.[1] But, aside from the motive, the firm's decision is reduced to valuing real assets. This valuation process is known as capital budgeting.

DEFINITION 45. ***Capital budgeting*** *is the process of valuing a real asset in a way that is consistent with the shareholders' wealth maximization goal as stated in Criterion 2.*

To keep a consistent framework for the following analysis, we will refer to the investment or the real asset that the firm is about to acquire as a **project**. In general, the firm might be in a situation of deciding among different projects. If the available projects are **independent** or unrelated, then the decision to accept one does not preclude the decision to consider the others. If, on the other hand, the projects are **mutually exclusive**, i.e., the available projects have or can perform the same function, then accepting one must eliminate the opportunity to accept all the others. In other words, in the latter case, the firm must be able to rank mutually exclusive projects, and thus, there ought to be some sort of criteria to do so. The decision criteria for capital budgeting will be discussed later. As for now, we will focus our attention on the relevant cash flows of any project.

A project with a **conventional cash flow pattern** consists of an initial outflow followed by a series of cash inflows. An **unconventional cash flow pattern**, on the other hand, consists of an initial outflow followed by a series of cash inflows and outflows. We will focus our attention on the former type of projects. In general, we can classify the different types of cash flows corresponding to any project into three categories, namely, **initial investment**, which we will denote by I_0, **operating cash inflow** at time t, which we will denote by OCF_t, and **terminal cash flow**, which we will denote by TCF.[2] The previous three types are formally defined as follows:

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[1] Other motives may include research and development, advertising, market studies and management consultancy.

DEFINITION 46. ***Initial investment*** *is the net cash outflow pertaining to a project at time $t = 0$. It is defined as*

$$\begin{aligned} I_0 &= \textit{initial investment needed to acquire a new asset} \\ &\quad - \textit{any after tax cash inflow from liquidation of old asset.} \end{aligned}$$

DEFINITION 47. *The* **operating cash inflows**, OCF, *are the after tax cash inflows that are expected to be generated from the project's operations over its life time. The cash flow at time t, CF_t, of a replacement project is the difference between the operating cash inflows from the new asset and the operating cash inflow from the old asset; that is,*

$$\begin{aligned} CF_t &= \textit{operating cash inflow from new asset at time } t \\ &\quad - \textit{operating cash inflow from old asset at time } t \\ &= \triangle OCF_t \end{aligned}$$

DEFINITION 48. *The* ***terminal cash flow*** *is the after tax non-operating cash flow occurring in the final year of a project and it is defined as*

$$\begin{aligned} TCF &= \textit{after tax cash flows from termination of new asset} \\ &\quad - \textit{after tax cash flows from termination of old asset.} \end{aligned}$$

2. Initial Investment

The initial investment is computed from Table 1. Perhaps the most important entries in the previous table that require further elaboration are the tax on sale of old asset and the net working capital. To determine the tax on sale of old asset, we require the sale price, SP, the initial purchase price, IP, and the book value, BV, which is defined by the equation:

$$BV = \text{installed cost of asset } - \text{ accumulated depreciation.}$$

The following is an example that illustrates the computation of the tax on sales of old asset.

EXAMPLE 42. *Suppose that ABC corporation, a small computer chip manufacturer acquired, two years ago, a machine tool with an installed cost of \$150,000. Suppose that the tool was being depreciated by 20% and 48% of*

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[2] Whether for expansion, replacement, or renewal, all projects have the first two types of cash flows. Not all projects, however, have the third type.

Installed Cost of New Asset		A
Cost of New Asset		
+ Installation Costs		
After-Tax Proceeds from Sale of Old Assets		B
Proceeds from Sale of Old Asset		
± Tax on Sales of Old Asset		
Change in Net Working Capital		C
$I_0 = A - B + C$		

TABLE 1. Determining initial investment

the installed cost in years 1 and 2 respectively. The book value of the asset today is

$$\begin{aligned} BV &= \textit{Installed cost - accumulated depreciation} \\ &= 150,000 - 30,000 - 72,000 \\ &= 48,000. \end{aligned}$$

Now, suppose ABC decided to sell the asset and replace it. How can we determine tax on sales of the old asset? The tax consequence will depend upon the sale price. In particular, the company could face one of the following four cases:

1. *Selling the asset at a price more than its initial purchase price*
2. *Selling the asset at a price more than its book value but less than its initial price.*
3. *Selling the asset at a price equal to its book value.*
4. *Selling the asset at a price less than its book value.*

In what follows, we explore each case in detail. In particular, in every case, we focus our attention on finding two values, namely, (1) the **capital gain**, which is the difference between the sale price and the purchase price, and (2) the **recaptured depreciation**, which is the portion of sale price that is above book value and below the initial purchase price.

CASE 1. ***Sale of an asset for more than its initial purchase price.*** *Suppose that everything is the same as in Example 42. Let income tax be 40%. Suppose that ABC sells the old asset for* \$160,000. *What is the tax on sale of such asset? In this case there is capital gain and recaptured depreciation and both are taxed as ordinary income. Notice that*

$$\begin{aligned} SP &= 160,000, \\ IP &= 150,000, \end{aligned}$$

and

$$BV = 48,000.$$

So, in this case,

$$SP > IP > BV.$$

The capital gain, by definition, is the difference between the sale price and purchase price; that is,

$$\begin{aligned} \text{Capital gain} &= SP - IP \\ &= 160,000 - 150,000 \\ &= 10,000. \end{aligned}$$

The recaptured depreciation is, by definition, the portion of SP *that is above* $BV = 48,000$ *and below* $IP = 150,000$*; that is,*

$$\text{Recaptured depreciation} = 150,000 - 48,000.$$

The diagram below shows the portions of the sale price that are considered capital gain and recaptured depreciation. Both incomes should be subject to corporate taxes.

$$SP = 160,000 \quad \underbrace{\overset{----}{\Downarrow}}_{\text{Capital Gain}} \quad IP = 150,000 \quad \underbrace{\overset{----}{\Downarrow}}_{\text{Recaptured Dep.}} \quad BV = 48,000$$

The table below summarizes the amount of tax that should be used in the computation of the initial investment.

Item	*Tax rate*	*Tax*
Capital Gain $= 160,000 - 150,000 = 10,000$	0.40	\$4,000
$+$***Recaptured Depreciation*** $= 150,000 - 48,000 = 102,000$	0.40	\$40,800
Total Amount of tax that goes in the calculation of I_0		\$44,800

CASE 2. ***Sale of an asset for more than its book value but less than its initial purchase price.*** *Suppose that ABC decided to sell its old asset for \$80,000. In this case, we have*

$$\begin{aligned} SP &= 80,000, \\ IP &= 150,000, \end{aligned}$$

and

$$BV = 48,000$$

It is obvious in this case that there is no capital gain. However, there is some realized gain in the form of recaptured depreciation. By definition, the recaptured depreciation is the portion of the sale price that is above BV *and below* IP*. But, since* SP *is already less than* IP*, then, the recaptured depreciation in this case is*

$$\text{Recaptured depreciation} = 80,000 - 48,000 = 32,000.$$

The diagram below shows the portions of the sale price that is considered recaptured depreciation. Notice that $SP < IP$ in this case.

$$IP = 150,000 \quad \_\_\_\_ \quad SP = 80,000 \quad \underbrace{\underset{\Downarrow}{\_\_\_\_}}_{\textit{Recaptured Dep.}} \quad BV = 48,000$$

Thus, the amount of tax that should be used in calculating the initial investment in this case is

$$32,000(0.40) = 12,800.$$

CASE 3. ***Sale of an asset for its book value.*** *Suppose the asset is sold at \$48,000. There is neither capital gain nor recaptured depreciation. Thus, there is no tax result for the sale of an asset at its book value, and hence, no effect on initial investment.*

CASE 4. ***Sale of an asset for less than its book value.*** *Suppose the asset is sold at \$38,000. In this case, the firm is incurring a loss equal to*

$$\textit{Loss on sale of the asset} = \$38,000 - \$48,000 = (\$10,000).$$

This loss will save the firm an amount of \$4,000 on taxes. This is the amount that would have been spent on taxes has the \$10,000 been gain instead of loss. The diagram below shows the loss on sale..

$$BV = 48,00 \quad \underbrace{\underset{\Downarrow}{\_\_\_\_}}_{\textit{Loss on sale.}} \quad SP = 38,000$$

The next relevant account that goes into the computation of initial investment is net working capital. The idea here is that when a firm decides to expand, usually the decision involves acquiring new asset(s), e.g., purchasing a new machine. This purchase will affect its cash, account receivables (A/R), and inventories. Therefore, when calculating initial investment, it is important to study the effect of expansion decisions on net working capital. First, we give a definition of net working capital, then we illustrate the computations by means of a simple example.

DEFINITION 49. *Net working capital (NWC) is the amount by which a firm's current assets (CA) exceeds its current liabilities (CL); that is,*

$$NWC = CA - CL.$$

Regardless of whether it is an increase or a decrease, it is not taxable because it merely involves an increase or decrease in current accounts.

EXAMPLE 43. *Consider Iron Heart, a manufacturer of metal products, that is currently expanding its operations. Table 2 summarizes the expected changes in the company's current accounts. In light of this information, we will analyze the effect on the company's initial investment. We begin by computing*

Current Account	**Change in Balance**
Cash	$+\$4,000$
Accounts Receivable (A/R)	$+\$10,000$
Inventories	$+\$8,000$
Accounts Payable (A/P)	$+\$7,000$
Accruals	$+\$2,000$

TABLE 2. Changes in Iront Heart's current accounts.

the change in net working, $\triangle NWC$*, which is simply the change in current assets minus the change in current liabilities; that is,*

$$\triangle NWC = \triangle CA - \triangle CL,$$

where

$$\triangle CA = \triangle cash + \triangle A/R + \triangle inventories = +\$4,000 + \$10,000 + \$8,000 = \$22,000,$$

and

$$\triangle CL = \triangle A/P + \triangle accruals = +\$7,000 + \$2,000 = \$9,000.$$

Thus,

$$\triangle NWC = \$22,000 - \$9,000 = +\$13,000.$$

Since any increase in cash, A/R*, and inventory is considered cash outflow, whereas any increases in* A/P *and accruals are considered cash inflows, then a positive change in* NWC *is considered a cash outflow whereas a negative change in* NWC *is considered a cash inflow. In our example,* $\triangle NWC$ *is positive. Therefore, there will be an increase in* NWC *as a result of this expansion decision, and hence, there will be an increase in initial investment.*

Now, we are ready to compute the initial investment.

PROBLEM 25. [3]*Powell Corporation, a manufacturer of aircraft, is trying to determine the initial investment required to replace an old machine with a new, more sophisticated model. The new machine's purchase price is* \$380,000 *and an additional* \$20,000 *is necessary to install it. It will be depreciated under straight line depreciation using a 5 year recovery period and a salvage value of* \$30,000. *The present (old) machine was purchased 3 years ago at a cost of* \$240,000 *and was being depreciated under a straight line depreciation using a 5 year recovery period and a salvage value of* \$40,000. *The firm has found a buyer willing to pay* \$280,000 *for the present machine and to remove it at his expense. The firm expects that a* \$35,000 *increase in CA and a* \$18,000 *increase in CL will accompany the replacement. Both ordinary income and capital gains are taxed at a rate of* 40%. *Calculate the firm's initial investment?*

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[3] This problem is adapted from Gitman (2006).

SOLUTION 25. *First, we calculate tax on sale of old assets. Depreciation of the machine (for only three years) can be calculated as follows:*

$$\frac{\$240,000 - \$40,000}{5} = \$40,000 \textit{ yearly.}$$

For 3 years, the depreciation shall be $\$40,000 \times 3 = \$120,000$. *The book value of the machine is*

$$\begin{aligned} BV &= \textit{installed cost - accumulated depreciation} \\ &= \$240,000 - \$120,000 = \$120,000. \end{aligned}$$

Notice here that

$$SP = 280,000,$$

$$IP = 240,000$$

and

$$BV = 120,000.$$

Since $SP > IP > BV$, *we are in Case 1, where the firm is selling the asset more than its initial price. Next, the amount of tax that should be used to calculate the initial investment is computed from the following table*

Item	***Amount***	*Tax rate*	***Tax***
Capital Gain $= 280,000 - 240,000 =$	40,000	0.40	\$16,000
$+$***Recaptured Depreciation*** $= 240,000 - 120,000 =$	120,000	0.40	\$48,000
Total Amount of tax that goes in the calculation of I_0			\$64,000

The change in net working capital is calculated as

$$\triangle NWC = \triangle CA - \triangle CL = \$35,000 - \$18,000 = +\$17,000.$$

Finally, the initial investment is computed from the following table:

A	***Installed Cost of New Asset***		400,000
	Cost of New Asset	*380,000*	
	+ *Installation Costs*	*20,000*	
B	***After-Tax Proceeds from Sale of Old Assets***		216,000
	Proceeds from Sale of Old Asset	*280,000*	
	± *Tax on Sales of Old Asset*	*(64,000)*	
C	***Change in Net Working Capital***		17,000
	$I_0 = A - B + C = 201,000$		

3. Operating Cash Flow

DEFINITION 50. *The operating cash inflow (OCF) is found by adding depreciation and any non-cash charges to the firm's net profit after tax (NPAT) from the company's income statement; that is,*

$$OCF = NPAT + \textit{Depreciation} + \textit{Other Non-Cash Charges}.$$

Item	**Amount**
Revenue	
− Expenses (excluding depreciation)	
Profits Before Depreciation and Taxes (PBDT)	
− Depreciation	
Net Profit Before Tax (NPBT)	
− Taxes	
NPAT	
+ Depreciation	
+ Other Non-Cash Charges	
Operating Cash flows (OCF)	

TABLE 3. Proforma income statement.

Year	Revenues and Expenses (excluding depreciation) with:			
	Proposed Machine		Present Machine	
	Revenues ($)	Expenses ($)	Revenues ($)	Expenses ($)
1	2,520,000	2,300,000	2,200,000	1,990,000
2	2,520,000	2,300,000	2,300,000	2,110,000
3	2,520,000	2,300,000	2,400,000	2,230,000
4	2,520,000	2,300,000	2,400,000	2,250,000
5	2,520,000	2,300,000	2,250,000	2,120,000

TABLE 4. Revenues and expenses (excluding depreciation) with the proposed and with the present machine.

The proforma income statement for a representative firm is shown in Table 3. Note that depreciation is subtracted at the beginning from the profit before depreciation and taxes to calculate taxes correctly. It is then added back to the net profit after tax because depreciation is considered as a source of fund.

PROBLEM 26. *Consider a corporation that is replacing an old machine with a new, more sophisticated model. The machine's purchase price is* $380,000 *and an additional* $20,000 *is necessary to install it. The proposed machine has an expected life of 5 years and is to be depreciated by* 20%, 32%, 19%, 12%, *and* 12% *in the first, second, third, fourth, and fifth year respectively. The present (old) machine was purchased 3 years ago at a cost of* $360,000 *and was being depreciated under a straight line depreciation using an 8 year recovery period and a salvage value of* $40,000. *The firm expects that a* $35,000 *increase in CA and a* $18,000 *increase in CL will accompany the replacement. Both ordinary income and capital gains are taxed at a rate of* 40%. *The company's estimates of its revenues and expenses (excluding depreciation) with and without the proposed new machine are given in Table 4. Notice that at the time the analysis was performed, the present machine was at the end of its third year of its cost recovery, and thus, it has 5 years*

Proposed Machine			
Year	Cost	%	Depreciation
1	400,000	20%	80,000
2	400,000	32%	128,000
3	400,000	19%	76,000
4	400,000	12%	48,000
5	400,000	12%	48,000
Total			\$380,000
Present Machine (remaining 5 years)			
Year	Cost	%	Depreciation
1			40,000
2			40,000
3			40,000
4			40,000
5			40,000
Total			\$200,000

TABLE 5. Depreciation for the proposed and the new machine.

of depreciation. Find the OCF in each year using the income statement format. Then use your findings to compute the change in OCF, which is taken to be the expected net cash flow, CF_t, over the life time of the machine.

SOLUTION 26. *First, we calculate the depreciation for the proposed and the new machine as shown in Table 5. Then, we compute the operating cash inflows over the life time of both the present and the proposed machine as shown in Table 6, and finally, we compute the change in operating cash inflows by taking the difference between the proposed and the present machine. The change in OCF is simply the expected net CF that is expected to be generated over the life time of this investment. For simplicity we will denote the change in OCF by CF.*

REMARK 14. *Notice that in replacement projects, although we didn't acquire the new asset yet, we consider both the OCF of the current asset and the expected OCF of the new asset over its life time. The point here is that we are contemplating replacing a machine, and thus, we wish to see the improvement in OCF as a result of this replacement if it happens.*

4. Terminal Cash Flow

DEFINITION 51. *The terminal cash flow (TCF) is computed from the equation*

$TCF = OCF$ *at terminal year* $+\triangle$ *in salvage value (new-old)*$+\triangle$ *in* NWC,

Proposed machine	Year 1 (time of analysis)	Year 2	Year 3	Year 4	Year 5
Revenue	2,520,000	2,520,000	2,520,000	2,520,000	2,520,000
− Expense	2,300,000	2,300,000	2,300,000	2,300,000	2,300,000
PBDT	220,000	220,000	220,000	220,000	220,000
− Depreciation	80,000	128,000	76,000	48,000	48,000
NPBT	140,000	92,000	144,000	172,000	172,000
− Taxes (40%)	56,000	36,800	57,600	68,800	68,800
NPAT	84,000	55,200	86,400	103,200	103,200
+ Depreciation	80,000	128,000	76,000	48,000	48,000
OCF ($*$)	164,000	183,200	162,200	151,200	151,200
Present machine	Remaining Y1	Y2	Y3	Y4	Y5
Revenue	2,200,000	2,200,000	2,400,000	2,400,000	2,250,000
− Expense	1,990,000	2,110,000	2,230,000	2,250,000	2,120,000
PBDT	210,000	190,000	170,000	150,000	130,000
− Depreciation	40,000	40,000	40,000	40,000	40,000
NPBT	170,000	150,000	130,000	110,000	90,000
− Taxes (40%)	68,000	60,000	52,000	44,000	36,000
NPAT	102,000	90,000	78,000	66,000	54,000
+ Depreciation	40,000	40,000	40,000	40,000	40,000
OCF ($\dagger$)	142,000	130,000	118,000	106,000	94,000
CF=$\triangle$ OCF $= * - \dagger$	22,000	53,200	44,200	45,200	57,200

TABLE 6. Annual operating cash flows for the proposed and the present machine. The change in operating cash inflows is the difference between the cash flows from the proposed and the present machine.

*where "**salvage value**" refers to the proceeds from sale of the new and the old asset, which is the amount expected upon termination of the project.*

Note that for replacement projects, proceeds from both the new assets and the old assets should be considered. For expansion projects, the proceeds from the old asset are simply zero.

PROBLEM 27. [4]*Robotics Corporation is considering replacing an existing hoist with one of two newer pieces of equipment: hoist A and hoist B. The existing hoist is 3 years old, cost \$32,000, and its useful life time is 5 years. It is depreciated under a straight-line depreciation method over 8 years recovery period. Hoist A costs \$40,000 to purchase and \$8000 to install. It has 5 years usable life and will be depreciated under a straight-line depreciation method using 5 years recovery period. Hoist B costs \$54,000 to purchase and*

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[4] Adapted from Gitman (2003).

	Profits before depreciation and taxes in ($)		
Year	with hoist A	with hoist B	with existing hoist
1	21,000	22,000	14,000
2	21,000	24,000	14,000
3	21,000	26,000	14,000
4	21,000	26,000	14,000
5	21,000	26,000	14,000

TABLE 7. Robotics' projected profits before depreciation and taxes for hoist A, B, and the existing hoist

\$6000 to install. It has also 5 years usable life and will be depreciated under a straight-line depreciation method using 5 years recovery period. Suppose the purchase of hoist A and hoist B would result, respectively, in a \$4,000 increase and a \$6,000 increase in net working capital. Assume that the projected profits before depreciation and taxes with each alternative hoist and the existing hoist are given in Table 7. The existing hoist can currently be sold for \$18,000 and will not incur any removal or other costs. At the end of 5 years, the existing hoist can be sold to net \$1,000 before taxes. Hoists A and B can be sold to net \$12,000 and \$20,000 before taxes, respectively, at the end of the 5-year period. The firm is subject to a 40% tax rate on both ordinary income and capital gain. (a) Calculate the initial investment associated with each alternative. (b) Calculate the incremental cash inflows associated with each alternative. (c) Calculate the terminal cash flow at the end of year 5 associated with each alternative.

SOLUTION 27. *We will only perform the analysis for hoist A. The analysis for hoist B is analogous and will be left as an exercise. First, note that the annual depreciation for hoist A is*

$$\frac{48,000 - 12,000}{5} = 7200 \textit{ yearly.}$$

The annual depreciation of the existing hoist is

$$\frac{32,000 - 1,000}{8} = 3875 \textit{ yearly.}$$

Thus, the depreciation of the existing hoist over 3 years is $\$11,625$ (3875×3). *The book value is* $\$20,375$ $(32,000 - 11625)$. *The selling price is* $\$18,000$ *and the initial price is* $\$32,000$. *Therefore, we are in case 4, where the firm is selling the asset by less that its book value. In this case, the firm is experiencing a loss of* $\$2,375$ $(20,375 - 18,000)$. *Consequently, the firm will save* $\$950$ *on taxes* $(\$3,375 \times 0.40)$. *The* $\$950$ *should be added to the proceeds from sale of old asset in order to obtain the after tax proceeds from sale of old assets. Finally, the change in net working capital is* $+\$4,000$. *Then, the* *initial investment is $\$33,050$ as shown from the table below.*

A	***Installed Cost of New Asset***		48,000
	Cost of New Asset	*40,000*	
	+ *Installation Costs*	*8,000*	
B	***After-Tax Proceeds from Sale of Old Assets***		18,950
	Proceeds from Sale of Old Asset	*18,000*	
	± *Tax on Sales of Old Asset*	+950	
C	***Change in Net Working Capital***		4,000
	$I_0 = A - B + C$	$= \$33,050$	

The incremental cash inflows associated with alternative A are computed from the table below:

With proposed machine	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*
$\triangle$ *PBDT (new-old)*	7,000	7,000	7,000	7,000	7,000
$-\triangle$ *Depreciation*	(3,325)	(3,325)	(3,325)	(3,325)	(3,325)
$\triangle$ *NPBT*	3675	3675	3675	3675	3675
$-\triangle$ *Taxes (40%)*	(1,470)	(1,470)	(1,470)	(1,470)	(1,470)
$\triangle$ *NPAT*	2,205	2,205	2,205	2,205	2,205
$+\triangle$ *Depreciation*	3,325	3,325	3,325	3,325	3,325
$CF \equiv \triangle OCF$	5,530	5,530	5,530	5,530	5,530

Finally, the terminal cash flow is computed as follows:

$$\begin{aligned} TCF &= OCF \text{ at terminal year} + \triangle \text{ in salvage value} + \triangle \text{ in } NWC \\ &= 5,530 + (12,000 - 1,000) + 4,000 \\ &= 20,530. \end{aligned}$$

Now, since we know how to compute I_0, CF_t, for $t = 1, ..., n$, where n is the life time of the project, and TCF, we are ready to actually evaluate the project.

5. Practice Problems

(1) Powell Corporation, a manufacturer of aircraft, is trying to determine the initial investment required to replace an old machine with a new, more sophisticated model. The new machine's purchase price is $\$380,000$ and an additional $\$20,000$ is necessary to install it. It will be depreciated under straight line depreciation using a 5 year recovery period and a salvage value of $\$30,000$. The present (old) machine was purchased 3 years ago at a cost of $\$240,000$ and was being depreciated under a straight line depreciation using a 5 year recovery period and a salvage value of $\$40,000$. The firm has found a buyer willing to pay $\$220,000$ for the present machine and to remove it at his expense. The firm expects that a $\$35,000$ increase in CA and a $\$18,000$ increase in CL will accompany the

replacement. Both ordinary income and capital gains are taxed at a rate of 40%. Calculate the firm's initial investment?

(2) Powell Corporation, a manufacturer of aircraft, is trying to determine the initial investment required to replace an old machine with a new, more sophisticated model. The new machine's purchase price is \$380, 000 and an additional \$20, 000 is necessary to install it. It will be depreciated under straight line depreciation using a 5 year recovery period and a salvage value of \$30, 000. The present (old) machine was purchased 3 years ago at a cost of \$240, 000 and was being depreciated under a straight line depreciation using a 5 year recovery period and a salvage value of \$40, 000. The firm has found a buyer willing to pay \$120, 000 for the present machine and to remove it at his expense. The firm expects that a \$35, 000 increase in CA and a \$18, 000 increase in CL will accompany the replacement. Both ordinary income and capital gains are taxed at a rate of 40%. Calculate the firm's initial investment?

(3) Powell Corporation, a manufacturer of aircraft, is trying to determine the initial investment required to replace an old machine with a new, more sophisticated model. The new machine's purchase price is \$380, 000 and an additional \$20, 000 is necessary to install it. It will be depreciated under straight line depreciation using a 5 year recovery period and a salvage value of \$30, 000. The present (old) machine was purchased 3 years ago at a cost of \$240, 000 and was being depreciated under a straight line depreciation using a 5 year recovery period and a salvage value of \$40, 000. The firm has found a buyer willing to pay \$100, 000 for the present machine and to remove it at his expense. The firm expects that a \$35, 000 increase in CA and a \$18, 000 increase in CL will accompany the replacement. Both ordinary income and capital gains are taxed at a rate of 40%. Calculate the firm's initial investment?

(4) Lombard company is contemplating the purchase of a new high-speed widget grinder to replace the existing grinder. The existing grinder was purchased 2 years ago at a cost of \$60,000; its was being depreciated under a straight-line depreciating method using a 5-year recovery period. The existing grinder is expected to have a usable life of 5 more years. The new grinder costs \$105,000 and requires \$5,000 in installation costs; it has a 5-year usable life and would be depreciated under straight-line depreciation with a 5-year recovery period. Lombard can currently sell the existing grinder for \$70,000 without incurring any removal or cleanup costs. To support the increased business resulting from purchase of the new grinder, accounts receivable would increase by \$40,000, inventories by \$30,000, and accounts payable by \$58,000. At the end of 5 years, the existing is expected to have a market value of zero; the new grinder would be sold to net \$29,000 after removal and cleanup

costs before taxes. The firm pays taxes at a rate of 40% on both ordinary income and capital gains. The estimated profits before depreciation and taxes over the 5 years for both the new and the existing grinder are shown in the following table.

Profits before depreciation and taxes in ($)		
Year	New grinder	Existing grinder
1	43,000	26,000
2	43,000	24,000
3	43,000	22,000
4	43,000	20,000
5	43,000	18,000

(a) Calculate the initial investment associated with the replacement of the existing grinder by the new one.
(b) Determine the incremental operating cash inflows associated with the proposed grinder replacement.
(c) Determine the terminal cash flow expected at the end of year 5 from the proposed grinder replacement.

(5) Do the analysis for Hoist B in Problem 27.

(6) Techno Corporation is considering replacing an existing hoist with a new hoist N. The existing hoist is 3 years old, cost $32,000, and its useful life time is 2 years. It is depreciated under a straight-line depreciation method over 5 years recovery period. Hoist N costs $54,000 to purchase and $6000 to install. It has 2 years usable life and will be depreciated under a straight-line depreciation method using 2 years recovery period. Suppose the purchase of hoist N would result in a $6,000 increase in net working capital. The projected profits before depreciation and taxes with hoist N and the existing hoist are given in the table below:

	Profits before depreciation and taxes in ($)	
Year	with hoist N	with existing hoist
1	22,000	14,000
2	24,000	14,000

The existing hoist can currently be sold for $18,000 and will not incur any removal or other costs. At the end of 2 years, the existing hoist can be sold to net $12,625 before taxes. Hoists N can be sold at $44,000 before taxes at the end of the 2-year period. The firm is subject to a 40% tax rate on both ordinary income and capital gain. The company can raise funds to finance the initial investment via 50% debt and 50% equity. The company's current bond is traded at a price $B = 800$. It has a face value $F = 1000$ and a coupon rate $c = 10\%$. The bond's maturity is 5 years. The company's current stock is traded at a price $P = 100$. Its expected dividends is $D_1 = 10$ and dividends are expected to grow at a constant rate $g = 16.4\%$.

(a) Calculate the initial investment in this replacement.
(b) Determine the incremental operating cash inflows associated with the proposed machine replacement.
(c) Compute the terminal cash flow associated the proposed machine replacement.

CHAPTER 7

Budgeting Techniques

There are three approaches/methods to value, and rank, investment projects. Namely, we will consider the **payback period** method, the **net present value** method, and the **internal rate of return** method. The projects considered are either mutually exclusive or independent. Mutually exclusive projects perform the same function, and therefore, selecting one implies rejecting or eliminating the other. Independent projects, on the other hand, are unrelated, and therefore, selecting one does not eliminate the other. The decision to accept independent projects is a matter of budget availability.

1. Payback Period

DEFINITION 52. *The* ***payback period*** *(PBP) is the number of years required to cover a project's cost.*

When investing in a project, management usually have an idea of how long it takes the project to cover its cost. Therefore, the decision is criterion to accept or reject a project is as follows: If the payback period is less than the maximum acceptable payback period, accept the project; otherwise, reject it.

EXAMPLE 44. *Suppose you have the following capital expenditure data of AAA corporation regarding a particular project: The projected initial investment is \$28,500 and the operating cash inflows are projected to be constant over the course of the next four years and are equal to \$10,000 annually. The payback period of this project is computed as follows:*

$$PBP = 0 \Longrightarrow -28,000 + 10,000 + 10,000 + 8,000 = 0.$$

Thus, the initial investment will be recovered completely after 2 years and 8/10 of a year, i.e., 2 years and $\frac{4}{5} \times 12 = 9.6$ *months. If the maximum acceptable PBP is 2.5 years, then the project should be rejected for its PBP is greater than the maximum acceptable PBP.*

The *PBP* is popular, easy to calculate, and considers cash flows rather than accounting profits. It can also be considered as a measure of risk exposure; the longer the firm must wait to recover its invested funds the higher the risk. It has, however, three main disadvantages: First, it does not take into consideration the time value of money for it does not discount the expected net cash flows of the project under consideration. Second, it does not consider the wealth maximization goal for it is not based on discounting cash flows to determine whether or not they add to the firm's value. Finally, it does not take into consideration the cash flows that occur after the PBP. The previous disadvantages call for valuation methods that are based on discounting cash flow, namely, the net present value method and the internal rate of return method.

2. DCF Methods: NPV & IRR

The DCF valuation methods treat an asset as a going concern and aims to find the present value of its **expected** future cash flows discounted back at a rate that reflects their riskiness. The roots of the DCF approach can be traced back to the development of the rule of present value and interest tables (see the brief historical survey in Chapter 5). We begin by discussing the NPV method.

DEFINITION 53. *The* ***net present value*** *of a project is found by subtracting the project's initial investment from the present value of its expected net cash flows discounted at an appropriate discount rate. This discount rate could be taken to be either the firm's cost of capital, or the opportunity cost of capital, or the project's RRR. In any case, the discount rate can be defined as the minimum return that must be earned on a project in order to leave the firm's market value unchanged. The* NPV *formulas is given by the equation:*

$$NPV = -I_0 + \frac{E\left[CF_1\right]}{(1+WACC)^1} + \frac{E\left[CF_2\right]}{(1+WACC)^2} + \cdots + \frac{E\left[CF_n\right]}{(1+WACC)^n}, \tag{7.1}$$

where $E\left[CF_t\right]$ *is the expected net cash flow in period* t, *for* $t = 1, ..., n$, *where* n *is the project's expected lifetime. In practice, the expected net cash flows are computed based on certain assumptions set forth by management regarding expected revenues and costs over the projected life time of the project. The discount rate of return is the firm's* $WACC$.

A $NPV > 0$ means that the present value of all discounted future cash flows exceeds the initial investment cost. This is indicative of a viable project. On the other hand, projects with negative NPV should be rejected. If two mutually exclusive projects with positive NPV are presented to management, the one with the higher positive NPV should be selected.

Notice that this method takes into account the time value of money, which is reflected in the use of a discount rate, which is the WACC. The NPV methods considers **all** expected future cash flows, and hence, is considered superior to the traditional PBP method.

Example 45. *Suppose you have the following information concerning two mutually exclusive projects: x and y:*

	Project x	*Project y*
Initial Investment (I_0)	\$28,500	\$27,000
Year\Operating Cash Flows		
1	\$10,000	\$11,000
2	\$10,000	\$10,000
3	\$10,000	\$9,000
4	\$10,000	\$8,000
WACC	10%	10%

The NPV for project x and y, denoted by NPV_x and NPV_y are computed as

$$NPV_x = -28500 + \frac{10000}{(1+0.10)^1} + \frac{10000}{(1+0.10)^2} + \frac{10000}{(1+0.10)^3} + \frac{10000}{(1+0.10)^4} = 3198.7,$$

and

$$NPV_y = -27000 + \frac{11000}{(1+0.10)^1} + \frac{10000}{(1+0.10)^2} + \frac{9000}{(1+0.10)^3} + \frac{8000}{(1+0.10)^4} = 3490.4$$

respectively. Since both values are positive and since $NPV_y > NPV_x$, then the firm should accept project y.

The second valuation method,which is closely related to the NPV method, is the IRR. The derivation of the IRR for a single investment was due to Boulding (1935), who argued that one can compute the IRR for a project by equating its initial investment cost to the present value of its expected cash flows. Boulding pointed out that the IRR of a single investment could be calculated from the following equation

$$I_0 = \frac{E\left[CF_1\right]}{(1+IRR)^1} + \frac{E\left[CF_2\right]}{(1+IRR)^2} + \cdots + \frac{E\left[CF_n\right]}{(1+IRR)^n}, \tag{7.2}$$

where I_0 is the initial investment, $E\left[CF_t\right]$ is period t's expected net cash flow (positive or negative) for $t = 1, ..., n$, where n is the total number of periods. Boulding argued that any rational investor should choose the project that maximizes his or her IRR. Boulding recognized that the solution of the previous equation is not mathematically simple and thought that a single solution could be found. He did not regard the existence of multiple internal rates of return as problematic.[1] Keynes (1936) gave the, now familiar and widely known, definition of IRR as the discount rate that makes the present value of the returns on an asset equal to its current price. Keyens referred to this definition of IRR as the **marginal efficiency of capital** and defined it as follows:

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[1] One of the drawback of the conventional IRR method is that it yields multiple rates of return, which could be ambiguous and confusing sometimes.

DEFINITION 54. *The internal rate of return on a project is defined as the discount rate that equates the present value of the project's expected future cash flows to the present value of the project's cost, i.e., its initial investment. Alternatively,* IRR *could be thought of as the rate of return that makes the project's* NPV *equal to zero; that is, if* $NPV = 0$, *then*

$$I_0 = \frac{E[CF_1]}{(1+WACC)^1} + \frac{E[CF_2]}{(1+WACC)^2} + \cdots + \frac{E[CF_n]}{(1+WACC)^n} \tag{7.3}$$

implies that $WACC = IRR$.

The previous definition suggests a connection between IRR and the $WACC$. Then, the decision criterion of whether to accept or reject projects based on IRR can be stated as follows: If the IRR of the potential project is greater than the firm's $WACC$, then the firm should accept the project as it will earn a rate of return greater than its cost. In this case, the project is expected to enhance the market value of the firm and add to the shareholders' wealth. This is precisely Criterion 2. Notice that a lower $WACC$ implies that the right-hand-side of equation (7.3) is greater than the left-hand-side, and hence, $NPV > 0$. On the other hand, if $NPV < 0$, then $IRR < WACC$, and thus, the project should be rejected.

The computation of the IRR is not straight forward. In practice, IRR can be computed by applying a sequence of trials and errors to equation (7.2). You can use a spreadsheet or the interpolation method when you narrow your guess. We have already showed how to compute the yield to maturity in a similar way using trial and error. In the following example, we illustrate the computation of IRR using trial and error and data interpolation. It is worth noting that the interpolation step is not essential. It can be replaced by continuous guessing using a spreadsheet.

EXAMPLE 46. *Suppose you have the following information regarding a given project:* $I_0 = \$12,000$, $E[CF_1] = \$5,000$, $E[CF_2] = \$5,000$, $E[CF_3] = \$8,000$, *and the firm's* $WACC$ *is* 15%. *Then,*

$$NPV = -12000 + \frac{5000}{(1+WACC)} + \frac{5000}{(1+WACC)^2} + \frac{8000}{(1+WACC)^3}$$

and the IRR *is the rate that satisfies the condition that* $NPV = 0$; *that is, the* IRR *satisfies the following equation:*

$$12000 = \frac{5,000}{(1+IRR)} + \frac{5,000}{(1+IRR)^2} + \frac{8,000}{(1+IRR)^3}.$$

To solve for IRR, *we begin by trying* $IRR = 20\%$ *(say). In this case, the right-hand-side of the previous equation is* 12,268.52, *which is slightly greater than* 12,000 *(the left-hand-side). So, we should try a higher rate to equate both sides. Let's try* $IRR = 25\%$. *In this case, the right-hand-side is* $11,296 < 12,000$. *Therefore, we conclude that the* IRR *has to be between* 20% *and* 25%. *We can continue, via trial and error, to plug in* *different values of* IRR *in the right-hand-side until we find the value that sets both sides equal, or alternatively, we can use interpolation to find a good approximation of the value of* IRR. *The interpolation yields the following equality:*

$$\frac{IRR - 20\%}{25\% - 20\%} = \frac{12,000 - 12,268.59}{11,296 - 12,268.59}.$$

This implies that $IRR \approx 21.38\%$.

Finally, it is worth mentioning that in case where future cash flows are expected to change sign, i.e., are expected to be positive and negative, multiple internal rates of return may result. In such a case, IRR can not be used for either accept/reject or ranking projects; the NPV should be used instead.

3. Practice Problems

(1) Lombard company is contemplating the purchase of a new high-speed widget grinder to replace the existing grinder. The existing grinder was purchased 2 years ago at a cost of \$60,000; its was being depreciated under a straight-line depreciating method using a 5-year recovery period. The existing grinder is expected to have a usable life of 5 more years. The new grinder costs \$105,000 and requires \$5,000 in installation costs; it has a 5-year usable life and would be depreciated under straight-line depreciation with a 5-year recovery period. Lombard can currently sell the existing grinder for \$70,000 without incurring any removal or cleanup costs. To support the increased business resulting from purchase of the new grinder, accounts receivable would increase by \$40,000, inventories by \$30,000, and accounts payable by \$58,000. At the end of 5 years, the existing is expected to have a market value of zero; the new grinder would be sold to net \$29,000 after removal and cleanup costs before taxes. The firm pays taxes at a rate of 40% on both ordinary income and capital gains. The estimated profits before depreciation and taxes over the 5 years for both the new and the existing grinder are shown in the following table. Assume that the WACC for Lombard company is 10%.

Profits before depreciation and taxes in (\$)		
Year	New grinder	Existing grinder
1	43,000	26,000
2	43,000	24,000
3	43,000	22,000
4	43,000	20,000
5	43,000	18,000

(a) Calculate the initial investment associated with the replacement of the existing grinder by the new one.

   (b) Determine the incremental operating cash inflows associated with the proposed grinder replacement.
   (c) Determine the terminal cash flow expected at the end of year 5 from the proposed grinder replacement
   (d) Calculate the IRR and the NPV of the proposed grinder.

(2) A machine currently in use was originally purchased 2 years ago for \$40,000. The machine is being depreciated under straight-line depreciation method and it has three years of usable life remaining. The current machine can be sold today to net of \$42,000. A new machine can be purchased at a price of \$140,000 and it has a 3 years usable life, it requires \$10,000 to install. The straight-line depreciation method is also used for the new machine. If the new machine was acquired, the investment in accounts receivable would be expected to rise by \$10,000, the inventory investment will increase by \$25,000 and accounts payable will increase by \$15,000. Profits before depreciation and taxes are expected to be \$70,000 for each of the next 3 years with the old machine and \$130,000 for each of the next 3 years with the new machine. At the end of the 3 years, the market price of the old and new machine will equal zero. The company pays taxes at a rate of 40%. The company can raise funds to finance the initial investment via 50% debt and 50% equity. The company's current bond is traded at a price $B = 800$. It has a face value $F = 1000$ and a coupon rate $c = 10\%$. The bond's maturity is 5 years. The company's current stock is traded at a price $P = 100$. Its expected dividends is $D_1 = 10$ and dividends are expected to grow at a constant rate $g = 16.4\%$. Answer the following questions:
   (a) Calculate the initial investment associated with the replacement of the existing machine by the new one.
   (b) Determine the incremental operating cash inflows associated with the proposed machine replacement.
   (c) Compute the terminal cash flow associated the proposed machine replacement.
   (d) Calculate the cost of debt before and after tax.
   (e) Calculate the cost of equity.
   (f) Calculate the WACC.
   (g) Evaluate whether the company should replace its old machine by the new one using the net present value method.
   (h) Define what it means by internal rate of return (IRR) and find the IRR of this project.

(3) Consider a bond with face value $F = \$1000$, coupon rate $c = 8\%$ paid quarterly, and maturity of 2 years. Answer the following questions.

(a) What is the current yield, capital gain, and the rate of return on this bond if you bought the bond today at a price 800 and decided to hold it till maturity?
(b) What is the yield to maturity if you decided to follow the same plan as in part a.
(c) What is the rate of return and the internal rate of return if you bought the bond today at a price 800 and sold it after one year at a price 1200? Notice here we can not use the yield to maturity since you are not holding it till maturity. The equivalent terminology is IRR and it is calculated by trial and error (you can not use the approximation method here).

(4) Techno Corporation is considering replacing an existing hoist with a new hoist N. The existing hoist is 3 years old, cost \$32,000, and its useful life time is 2 years. It is depreciated under a straight-line depreciation method over 5 years recovery period. Hoist N costs \$54,000 to purchase and \$6000 to install. It has 2 years usable life and will be depreciated under a straight-line depreciation method using 2 years recovery period. Suppose the purchase of hoist N would result in a \$6,000 increase in net working capital. The projected profits before depreciation and taxes with hoist N and the existing hoist are given in the following table:

		Profits before depreciation and taxes in (\$)	
Year		with hoist N	with existing hoist
1		$22,000$	$14,000$
2		$24,000$	$14,000$

The existing hoist can currently be sold for \$18,000 and will not incur any removal or other costs. At the end of 2 years, the existing hoist can be sold to net \$12,625 before taxes. Hoists N can be sold at \$44,000 before taxes at the end of the 2-year period. The firm is subject to a 40% tax rate on both ordinary income and capital gain. The company can raise funds to finance the initial investment via 50% debt and 50% equity. The company's current bond is traded at a price $B = 800$. It has a face value $F = 1000$ and a coupon rate $c = 10\%$. The bond's maturity is 5 years. The company's current stock is traded at a price $P = 100$. Its expected dividends is $D_1 = 10$ and dividends are expected to grow at a constant rate $g = 16.4\%$. Answer the following questions:

(a) Calculate the initial investment in this replacement.
(b) Determine the incremental operating cash inflows associated with the proposed machine replacement.
(c) Compute the terminal cash flow associated the proposed machine replacement.
(d) Evaluate whether the company should replace its old machine by the new one using the net present value method.

CHAPTER 8

Financial Evaluation in Practice

1. Financial Evaluation: Sources of Information

In the previous chapters, we have seen how the general DCF formula is applied to assess the value of any real or financial asset. In particular, we have learned how can we find the fundamental price (or the present value) of a corporate bond that promises a fixed payment over a specific period of time. Recall that the discount rate that was used in the DCF formula for bonds was the bank rate. We have also learned how to find the fundamental value of a corporate share using different assumptions about the growth rate of dividends. The discount rate, in this case, was taken to be the expected rate of return on an similar asset in the same risk class. With respect to real assets, we have also learned how to compute the initial investment, operating cash flows, and the terminal cash flow pertaining to replacing or acquiring a machine or a production equipment. The discount rate in the DCF formula, in this case, was the firm's WACC. In this chapter, we will discuss how all the previous notions can be applied to assess the financial value of an investment project in practice. This process is known as financial evaluation.

A **financial evaluation** (also known as **feasibility study**) is a study of the viability of a project from the investor's point of view. Since investors are always after profit/wealth maximization, financial evaluation looks at direct revenues and direct costs of the project. Any indirect effects or externalities are not considered in the financial evaluation. For instance, if the project has indirect positive or negative effects on the economy, the effect of these externalities is not included in the financial evaluation; rather it is included in the so-called cost-benefit analysis of the project. Thus, financial evaluation is different than cost-benefit analysis. The former is used by investors who wish to maximize profit. The latter, however, is used by regulators and policy makers to assess the effect of a project on the economy. In this chapter, we focus on the former rather than the latter.

Two sets of information are needed for financial evaluation (financial feasibility study) of a project; namely, (1) **market information** and (2) **technical information.** Market information includes data or information on the market size, market share, prices, demand elasticity of product, and other market related information. Technical information includes information on the technology used in production, the productivity of labor, the required working hours, and other technical related information. In the following sub-sections, we briefly explore these sources of information.

1.1. Market Information. Market information are obtained by analyzing the market structure in which the project is expected to operate. Economists classify market structures into four types: (1) perfect competition, (2) monopoly, (3) monopolistic competition, and (4) oligopoly. Perfect competition is considered as a benchmark structure. In other words, it is the **ideal** market that we as economists would like to see. Unfortunately, this market is too ideal so that it does not exist in real life. Being a hypothetical market though does not lessen its importance as a benchmark against which the performance of all other distorted markets can be measured. By distortion here I mean either **quantity distortion** or **price distortion** or both. A distorted quantity is a quantity above or below the equilibrium price. Similarly, a distorted price is a price that is above or below the equilibrium price. The price of a particular product and the corresponding quantity resulting from the hypothetical perfect competitive market are considered the equilibrium price and the equilibrium output, respectively, of that particular product. All prices and output levels resulting from any market structure other than perfect competition are distorted and the extent of distortion can be measured by contrasting these prices and quantities to their perfect competitive counterparts. For instance, if the perfect competitive price of a product is x and the corresponding monopoly price $y > x$, then the price distortion caused by the monopolist is simply the difference $y - x$.

While, on one hand, the perfect competitive market is the most efficient and ideal market, on the other hand, the monopoly market is the least efficient and most distorted. A monopolist will always produce a quantity smaller than the perfect competitive output and will always charge a price higher than the perfect competitive price and, thus, causes both: quantity distortion and price distortion in the market. That's why some governments (including Canada) prohibit the formation of monopoly or cartels in the market. The following is a brief discussion on the characteristics of each market structure.

1.1.1. *Perfect Competition.* A perfect competitive market is a hypothetical structure built on the following assumptions:

(1) There are many buyers and many sellers operating in such a market.
(2) All firms sell homogeneous (exactly identical) products, i.e., the products sold have the same shape, same size, same color, same weight, and so on. Obviously, this assumption is not a realistic assumption. In reality, it is hard to find two identical products. The closest example would be **agricultural products**.

(3) Producers in perfectly competitive markets don't have any market power. Since all producers in this market are selling homogeneous products, it follows that they are all **price takers**, i.e., they have no control over the market price. In business or economic terms, price control is referred to as "**market power**." Thus, in perfect competition, producers have no market power. The market price is determined by the interaction between the market supply and the market demand for the product. The resulting price is the equilibrium market price and all producers take that price as is.

(4) Perfect competitive markets are characterized by **perfect information**; that is, consumers know the nature of the product sold and the price charged by each firm.

(5) Each firm's output is small relative to the size of the industry's output.

(6) Free entry to and exit from the market. This means that are no barriers to enter or exit this market. Examples of barriers are monopolistic power and economies of scale.

1.1.2. *Monopoly.* The second market structure is Monopoly. In this market there is only a single producer selling a single product that does not have close substitutes. Since the monopolist is the single producer, obviously he or she has full control over price if not regulated. As opposed to perfect competition, there are barriers to enter a market dominated by a monopolist. Having full market power, the monopolist can reduce the market price below the marginal cost of production of the potential entrant and, thus, preventing the latter from entering the market. It is worth mentioning that, in general, unregulated monopolies rarely persist over long periods because of the innovation of new production processes and new products in the long-run. However, a monopoly can continue to survive as a result of owning a patent or using predatory pricing.

Sometimes the nature of the industry itself forces a monopoly behavior. This is the case of a **natural monopoly**, where the average cost of producing any output declines with the scale of operations, i.e., the whole market might be supplied at least cost by a single firm. Usually this firm (the natural monopolist) is owned by the government. Trains, electricity, and utilities companies are all examples of natural monopolists.

An unregulated monopolist can exercise price discrimination among consumers by charging different prices to different units sold of the same products or different prices to the same number of units sold of the same products for reasons not associated with differences in cost. Basically, the price differences do not reflect the differences in costs, rather they reflect differences in buyer's valuations of the product. The first type of discrimination (charging different prices to different units sold of the same product) is known as **perfect discrimination**. In practice, it is difficult to achieve this type of discrimination as the monopolist must be able to detect the willingness to pay for each consumer. In theory, if this is possible, i.e., if the monopolist is able to detect the willingness to pay (the demand curve) of his or her consumers, the monopolist can extract all of the consumer surplus by charging the maximum amount that they are willing to pay and thus, achieving a supernormal profit. In practice however, it is difficult to accurately estimate the willingness to pay of each customer. Although, perfect discrimination is very difficult to implement in reality, monopolists can still achieve the outcome realized under perfect discrimination by implementing a **two-part tariff pricing scheme**, where the consumers are charged a lump sum fee to enjoy the service provided by the monopolist and a price per unit consumed. The monopolist will set the fee equal to the consumer surplus and the price charged per unit equal to his/her marginal cost of production. Examples of two-part tariff schemes implemented by monopolists are Disney Land in the United States and Wonderland in Canada. The second type of discrimination (charging different prices to the same number of units sold of the same product) is commonly found in practice. Consider, for instance, the public transport agency in Montreal, Quebec, known as The Societe de Transport de Montreal (STM), which operates transit bus and rapid transit services in Montreal. This natural monopolist offers lower priced tickets to students and, thus, charges different prices to the same number of units sold of the same product.

To sum up, as a market structure, monopoly is based on the following assumptions:

(1) One single producer.
(2) The monopolist sells one single product that does not have close substitutes.
(3) The monopolist has full control over price if not regulated.
(4) There are barriers to enter such a market.

1.1.3. *Monopolistic Competition.* The third market structure is monopolistic competition. This structure can be seen as a combination of monopoly and perfect competition; hence, its name. In this market structure, the industry, like perfect competition, is characterized by **many buyers and many sellers,** i.e., large number of small firms. The other feature which is similar to the monopoly structure is that the products sold in this market are **highly differentiated**, which implies that each firm acts as a monopolist in its own segment. Each firm capitalizes on its competitive advantage, i.e., each distinguishes itself from others by focusing on certain characteristics of its product to gain a competitive edge in the market. This strategy leads to the establishment of brand names and advertising. This, in turn, gives the firm a degree of market power in its own market segment. This is the type of market that we experience every day. Examples of monopolistic competitive industries are clothing, cars, furniture, groceries, fast food, coffee, and so on.

The barriers to enter such a market are low. This, in turn, implies that competition is fierce and, thus, long-run equilibrium in such a market gives rise to **excess capacity**. The idea here is that in the long-run existing firms in a monopolistic competitive market will produce an output below their maximum capacity. This is due to the fact that there are low barriers to enter such a market so that when more firms enter in the long-run, the number of customers that each existing firm has will decrease, which, in turn, will result in excess capacity.

As a market structure, monopolistic competition is based on the following assumptions:

(1) The industry is characterized by many buyers and many sellers (large number of small firms).
(2) Firms sell highly differentiated products. Each firm chooses the characteristics to give to the products that it will sell. Each firm produces a specific brand from the industry's highly differentiated products. Therefore, we can say that this product differentiation assumption leads the establishment of brand names and advertising, which gives firms a degree of market power in their own market segments.
(3) Firms have a degree of market power.
(4) All firms have access to the same technology and, therefore, have same cost curves.
(5) Free entry to and free exit from the market (similar to the PC market)

1.1.4. *Oligopoly Market.* The last market structure is the oligopoly market, which is characterized by a few number of producers (two or more firms, at least one of which has a large market share). Usually the number of firms varies between 2 to 10 or 13 firms at most. The most important feature of such a market is that all the existing firms are producing **slightly differentiated** products, which makes the follower firms' strategies contingent on the leaders' strategies. Due to this type of leader-follower setup, game theory is at the core of oligopoly market. Discussing game theory is an interesting topic by itself but, unfortunately, it is beyond the scope of this discussion. In general, firms operating in an oligopoly market can follow one of two strategies: either (a) cooperative strategy (collusion) or (b) non-cooperative strategy (competition). If they **collude**, they will form a monopoly structure, which of course is illegal, and thus explicit collusion is unlikely to happen. So, collusion is most likely **implicit** (tacit collusion). Since it is implicit collusion, each firm will have a tendency to deviate from

Market Structure	**Number of Firms**	**Types of Products**	**Price Control**	**Entry Barriers**	**Exam**
PC*	Many	homogeneous	no	free entry	Agricu
Monopoly	1	single	full	high	Micros
Duopoly	2	slightly differentiated	some	high	Pepsi &
Oligopoly	3-13	slightly differentiated	some	high	Steel
MC**	Many	highly differentiated	little	low	Clothi
* Perfect Competition					
** Monopolistic Competition					

TABLE 1. Characteristics of various market structures.

the collusion agreement (cheat) to make more profit. The cheater will be tempted to sell more at the high collusion price and, thus, make more profit. Therefore, we can say the outcome of collusion, if it happens, will **not** be **sustainable** because of **cheating**. One way to make the outcome sustainable is to impose a **credible threat** in case of cheating. On the other hand, if the firms decide to follow a non-cooperative strategy, they will compete either in terms of prices or in terms of quantities. Whatever the type of competition is, firms will have to play the oligopoly game!

To sum up, as a market structure, oligopoly is based on the following assumptions:

(1) Few number of producers (two or more firms, at least one of which has a large market share). Because this matter is characterized by small firms, each firm's decision is contingent on the other competitors' decision.
(2) The products sold in the market are slightly differentiated.
(3) Producers have a degree of price control.
(4) There is barriers to enter such a market.

Table 1 summarizes the characteristics of all the previous types of market structures.

1.2. Technical Information. Technical information refers to relevant data pertaining to the location, capacity, costs, and revenues of the project. Perhaps, the most crucial part of technical information is the calculation of the project's market share. There are many ways to project/forecast the market share of a firm as a potential entrant into an industry. The classical approach is to find the total revenue of the entire industry, the main players and their share, and then divide the remaining shares among the other players (including your firm as a potential entrant). There are of course variations to this strategy depending on the nature of the product and the market structure.

It is also useful sometimes to collect information on similar firms in the same risk class. If the potential firm is projected to sell a product that is similar to one provided by another firm in a monopolistic competitive industry, then studying this firm and figuring out its market share is considered a good proxy. At any rate, critical thinking is definitely required in this stage.

Acquiring financial information on a currently operating business could be done using the service of third party **financial data vendors.** These companies make a living by providing financial information to different players in the financial system. Examples of these companies are **Bloomberg**, **Thompson Reuters**, **Fact Set**, **Markit**, **Morningstar**, **S&P Capital IQ**, **SIX Financial Information,** and many others. Nowadays, an abundance of information is available for free from **financial websites** such as **Google Finance** and **Yahoo Finance**.

There are many avenues a financial analyst can pursue when shopping for information. In general, information on publicly traded corporations can come from one of two different sources: **primary sources** or **secondary sources**. The former sources include information and data from the **annual and quarterly reports** prepared publicly traded corporations as well as from the **public conferences calls** hosted by these corporations after the publication of their reports. The latter sources include data provided from third parties, e.g., Bloomberg and Standard & Poor, sell-side analysts' reports, and industry reports. Below is a brief discussion on both sources.

Annual and Quarterly Reports are among the primary sources of data for financial analysts. **Annual Reports** are prepared by the company management for the purpose of providing investors and other stakeholders information on the company's operations, financial numbers and business conditions. Some of these reports are mandatory, i.e., enforced by regulators, whereas others are prepared solely for the shareholders of the corporation. The purpose of the mandatory reports is to ensure full disclosure and transparency of information. This results in a healthy investment environment; a well informed investor tends to make rational investment decisions, which, in turn, guarantees the existence of an efficient capital market. For corporations listed in the United States, mandatory Annual Reports enforced by U.S. regulators are called **10-K Reports**. For those listed in Canada, mandatory reports are called **Annual Information Forms**, or **AIF** for short. For companies cross-listed on both Canadian and U.S. exchanges, the regulatory form used is called **40-F**. In addition to the 10-K or AIF Reports, most corporations often prepare **Annual Reports** for their owners (shareholders). Those reports contain a **letter from the firm's Chief Executive Officer** or Chairman, a **Management Discussion & Analysis** (MD&A) section, and a **full set of financial statements**, i.e., balance sheet, income statement, and cash flow statements.[1] A **Quarterly Report** is just a detailed version of an Annual Report. In the U.S., a Quarterly Report is called **10-Q** Report. In Canada, there is no equivalent of a Quarterly Annual Information Form. Usually the Quarterly Report is much shorter and contains less detailed discussions on the business conditions or operations. Financial disclosure in Quarterly Reports is less thorough than the disclosure in Annual Reports. For instance, an Income Statement in a Quarterly Report might display depreciation expense as part of the cost of goods sold and not as a separate entity as in the case of a detailed Annual Report.

In addition to Annual and Quarterly Reports, the management of public companies usually hosts a **Public Conference Call** after reporting the company's quarterly results. During the conference, the management presents its own remarks on the quarterly results and then holds a question & answer session for the public. Management's remarks involves its own assessment of the company, the trends in the industry, and the expected future outlook of the company, e.g., future growth plans. The management's remarks and statements given during these conferences provide investors and analysts some sort of insight regarding the effectiveness of the corporation, its financial soundness, and the transparency of the management team in acknowledging the challenges that the company is facing and the actions taken to overcome these challenges. In addition to the management's remarks, the question & answer session is of crucial importance as it allows sell-side analysts, e.g., those who are employed by broker-dealers such as Goldman Sachs or Morgan Stanley, to ask the Chief Financial Officer of the company questions regarding the company's finances or operations. By listening to their answers, financial analysts gain further insight regarding the company's prospective plans and, consequently, adjust their financial model of the company accordingly. That's why public conferences are considered a valuable source of information for investors and financial analysts. Conference calls are usually scheduled on the same day as when the company reports its quarterly results. The public can call in and listen to the conference call using the phone number published in the press release. Some companies offer a webcast version of the conference call on their investor relations webpage.[2]

As for the secondary sources of information, third party providers, such as **Bloomberg**, **Thomson First Call**, and **Standard & Poor,** provide consensus forecasts for public companies. These consensus predictions include forecasts of the company's earning per share and revenue in the next fiscal year. Depending on the company, consensus numbers may include forecasts for gross margins, operating margins, production, effective tax rates, interest expense, and so on. Furthermore, third party providers offer downloadable financial statements in Excel format and other financial ratios that may be of interest to the financial analyst. It is worth mentioning that sometimes the calculations of financial ratios differ from one provider to the other, e.g., using ending inventory versus average inventory in the computation of inventory turnover ratio, and, consequently, it is paramount for the analysts who are using these ratios to verify how they have been calculated.

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[1] In order to reduce the cost of disclosure, some corporations issue a more detailed 10-K or AIF designed to meet the regulatory disclosure requirements and, at the same time, provides the necessary information for shareholders.

[2] Only limited companies provide free-transcript of their conference calls but Bloomberg users can find the transcript of the quarterly conference calls under the <**EVT**> security function.

Another type of secondary sources of information is sell-side reports, which can be either on an industry or on a specific company. These reports are particularly useful for financial analysts who are new to the industry under consideration. Although these reports are useful, they must be treated with caution. The analyst should not put too much emphasis on the final investment recommendations that these reports provide; the focus, however, should be on the key **drivers** mentioned in the report. These drivers are considered useful for the analysts when they formulate their own models. Also, it is worth noting that sell-side analysts frequently write general **industry reports** on the industry they cover. These reports contain information on the latest trends in the industry under consideration, the major players, and the market structure. Industry reports are considered another valuable secondary source of information. Below is a list of general industry reports:

- Canadian Association of Petroleum Producers for oil and gas data.
- Energy Information Agency for energy/oil and gas data.
- World Gold Council for gold mining data.
- Office of the Superintendent of Financial Institutions (OSFI), Basel Committee on Banking Supervision, and the Federal Reserve for data on banking.
- Statistics Canada (Statscan) and U.S. Census Bureau for consumer related data for retail companies.
- Mining Journal for mining data.
- Canadian Real State Association (CREA) for real estate.

As for information pertaining to a particular industry, **economic data**, which is also known as **macro data**, could be useful for it includes information about the state of the economy such as GDP, interest rates, money supply, unemployment, and other economic indicators. Primary sources of economic data include central banks, e.g., **Bank of Canada**, as well as numerous **economic bureaus** and **research institutes**, better known as **think-tanks**, e.g. **Statistics Canada**. Like financial data, some economic data is only available at regular intervals through the year, e.g., quarterly or semi-annually. For instance, the national GDP in Canada is published quarterly.

2. Financial Evaluation: The Methodology

In financial analysis, we differentiate between two concepts: financial evaluation and asset valuation. Financial evaluation of a project means testing its financial viability, i.e., assessing the expected performance of the project in terms of the return on investment that it could generates. Financial evaluation means obtaining the value of the project as a going concern in the market; that is, how much the project is worth or how much investors are willing to pay for this project today. It is worth noting that obtaining the value of a project as a going concern in the market is totally different than just valuing assets. If you are seeking the value of the assets of a project, you valuate the project's assets one by one independently. This is not valuation of a project; rather, it is liquidation. When you evaluate the project as a going concern, you are valuing the project as one unit, and thus, you are valuing the capability of all project's assets. There are five steps to conduct a proper financial analysis of a project:

1. Analyzing revenues and costs
2. Estimating the capital required to finance the project.
3. Finding the proper financial structure
4. Performing financial evaluation
5. Performing sensitivity analysis

In what follows, we discuss each step in detail.

3. Step 1: Analyzing Revenues and Costs

Since a project, unlike any asset, has no expected life, we calculate our projection of the revenues and costs of the project for a certain period of time and capture the rest through the residual value of the project. In practice, a five-year projection period is sensible.

3.1. Revenues. Revenues are defined by the equation

$$\text{Revenue} = P \times Q,$$

where Q is the total physical quantity produced and P is the market price per unit. For industrial projects, we consider quantities and prices. Yet, in case of services, we consider the value of the service since we can not differentiate between P and Q for a service. As mentioned above, any indirect revenues should not be included in this analysis.

Quantities sold or services provided can be estimated of projected over the life time of the project based on the estimated market share or the estimated demand on the product or service offered. Usually, we project the first year revenues and assume a sensible growth rate over the remaining life time of the project.

3.2. Costs of Operations (flow of costs of a running business). Costs of operations can be divided into two categories: (a) direct costs and (b) indirect costs. A direct cost is a cost that is directly related to production. Direct costs are always considered as variable costs. Cost of raw materials and wage rate are examples of direct costs. Indirect costs have nonlinear relationship with production. Indirect costs can be subdivided into variable indirect costs such as costs of utilities and marketing costs and fixed indirect costs such as salaries of top management.

3.3. Proforma Income Statement. After breaking down the revenues and costs, we summarize them in what is known as the proforma *income statement as follows:*

	Revenues
less	Direct Costs
	Gross Profit
less	Indirect Costs
	Gross Operating Profit (GOP)

GOP is a significant profit parameter as it measures the capability of the project to make profit. For any business, GOP should range from a maximum of 40% to a minimum of 25% for a sensible range.

4. Step 2: Capital Required to Finance a Project

Assets are divided depending upon the purpose of buying them into two groups: (a) fixed assets and (b) current assets. If the purpose of buying the asset is to be used in producing other asset, the asset is considered a fixed asset. Land, building, and equipment are examples. The capital required to finance fixed assets is called fixed capital. If the purpose of buying the asset is to resell it, such as raw materials, the asset is considered a current asset and the capital required to finance current assets is called the working capital.

The total investment cost, TIC, is defined as the sum of the fixed capital, FC, and the net working capital, NWC; that is,

$$TIC = FC + NWC,$$

where FC represents the resources required for constructing a project. In particular,

$$FC = \text{Fixed Investment} + \text{Pre-production Capital Costs}.$$

The NWC represents the resources needed to operate the project. It is the difference between current assets, CA, and current liabilities, CL; that is,

$$NWC = CA - CL.$$

It is worth noting that in the pre-investment stage, two mistakes are frequently made: (1) the working capital is either not included at all or at insufficient amounts, which causes serious liquidity problems to the project, and (2) the TIC is sometimes confused with total assets. As mentioned above,

$$\begin{aligned} TIC &= FC + NWC \\ &= \text{Fixed Investment} + \text{Pre-production Capital Costs} + CA - CL, \end{aligned}$$

whereas

$$Total\ Assets = Fixed\ Assets + \text{Pre-production Capital Costs} + CA.$$

4.1. Fixed Capital (FC). This is the capital required to finance fixed assets. In general, fixed assets consists of (1) **fixed investments** (tangible assets) and (2) **pre-production capital cost** (intangible).

Fixed investments should include the following:

- **(a):** Land and site preparation
- **(b):** Buildings and civil work
- **(c):** Plant machinery and equipment
- **(d):** Certain incorporated fixed assets such as industrial property rights

Pre-production capital costs are incurred prior to commercial production. These expenditures usually take place during the various stages of project formulation and implementation. They include the following:

- **(a):** Expenditures incurred during the registration and formation of the company including legal fees and documentation
- **(b):** Expenditures for preparatory studies. These are 3 types
  - **(i):** Expenditures on pre-feasibility, feasibility studies of the project
  - **(ii):** consultation fees
  - **(iii):** Other expenses for planning the project

(c): Pre-production expenditures including salaries, travel expenses, training costs, and so on.

4.2. Net Working Capital (NWC). Working capital is the capital required to finance current assets. NWC is defined as current assets, CA, minus current liabilities, CL; that is,

$$NWC = CA - CL.$$

Note that CA consists of cash, accounts receivables (A/R), inventories (raw materials, supplies, packaging materials, spare and small tools), work in progress, and finished products, whereas CL consist mainly of accounts payable (A/P).

Working capital finances the time lag between two actions. In general, a business has four gaps that need to be financed:

(1) **Gap between sales and cash collection**: The cash required to finance the difference between sales and cash collection is called **accounts receivable** (A/R).

(2) **Gap between production and sales**: During this period, you have to pay for raw materials, wages, and other overhead costs. The cash required to finance the gap between production and sales is called **inventory of final products**.

(3) **Gap between buying raw materials and using them in business**: Usually raw materials are purchased and kept in stock till being used in production. The cash required to finance the gap between buying raw materials and using them in business is called **inventory of raw materials**.

The **Gross Working Capital** is simply the sum of (1), (2), and (3). Sometimes you have payment allowance which need to be taken into consideration: this is our fourth gap.

(4) **Gap between buying raw materials and actually paying the amount due**: The capital saved is called **accounts payable** (A/P). Therefore, the NWC is calculated as

$$NWC = Gross\ Working\ Capital - A/P$$

The following example illustrates how NWC is computed.

EXAMPLE 47. *Suppose you have the following income statement of a given firm, where the cost of raw materials is \$600 and the total cost of operations is \$900.*

Revenue			$a = 1500$
	Cost of Raw Materials	*600*	
	Other costs	*300*	
Total Costs of Operations			$b = 900$
NPBT			$a - b = 600$

Assume further that the you have the following data.

	Coverage Period	**TOC***	$Working\ Capital = \frac{Annual\ cost\ of\ operations}{TOC}$
A/R	3 months	$\frac{12}{3} = 4$	$\frac{Annual\ cost\ of\ operations}{TOC} = \frac{900}{4} = 225$
Inventory of Final Product	2 month	$\frac{12}{2} = 6$	$\frac{Annual\ cost\ of\ operations}{TOC} = \frac{900}{6} = 150$
Inventory of Raw Materials	3 months	$\frac{12}{3} = 4$	$\frac{Annual\ cost\ of\ raw\ materials}{TOC} = \frac{600}{4} = 150$
-			
A/P	1 months	$\frac{12}{1} = 12$	$\frac{Annual\ cost\ of\ raw\ materials}{TOC} = \frac{600}{12} = 50$
NWC			475
*TOC refers to turnover coefficient			

The working capital is calculated as shown in the last column of the table above. Once you estimate the capital required; that is the fixed capital and the net working capital, the next step is to find the proper financial structure.

5. Step 3: Proper Financial Structure

There are two sources of finance for any project: (a) Equity financing and (b) loan financing. The general financial pattern for an industrial project is to cover the initial capital requirement by equity and long-term loans and to meet the working capital requirements by additional short and medium term loans from national banks. Yet, various permutations exist for this framework.

We need to determine the optimal amount of the loan from both: the investor point of view and the from the bank manager point of view. The idea here is that, from an investor's point of view, as long as the cost of the outside source of finance (cost of bank loan (i)) is less than the cost of the investor's equity (cost of his/her money), then the investor would go and get 100% of the capital required from the bank.

We are after finding out the maximum amount that the investor can borrow from the bank that will achieve his/her objective, which is profit maximization of course and, on the other hand, ensures the availability of liquidity for the bank, i.e., achieving the bank manager's objective by making sure that the amount of the loan will be repaid to the bank. In other words, we are after finding a formula that achieves both objectives: profit maximization from the point of view of the investor and liquidity from the point of view of the bank manager. Let

I = investment cost

r = return on investment

L = loan

i_b = interest rate on borrowings

R = repayment of the loan

s = number of installments

Then, in order to achieve both objectives, we ought to have

$$Profit - interest\ charges = repayment\ of\ the\ loan, \tag{8.1}$$

or equivalently,

$$(I \times r) - (L \times i_b) = R,$$

where

$$L = R \times s. \tag{8.2}$$

Now, consider equation (8.1) and multiply both sides by $1/R$. This yields

$$\frac{(I \times r)}{R} - \frac{(L \times i_b)}{R} = 1, \tag{8.3}$$

and from equation (8.2), we have

$$s = \frac{L}{R}. \tag{8.4}$$

Plug (8.4) in (8.3) and solve for R. This gives

$$R = \frac{I \times r}{1 + (s \times i_b)} \tag{8.5}$$

and

$$L = R \times s. \tag{8.6}$$

Equations (8.5) and (8.6) give the optimal repayment of the loan and the optimal loan respectively. Notice, from (8.5), that the higher the return on investment, r, the higher the repayment of the loan, R, and the loan, L. This means that a profitable business will always be credible to receive loans. Also notice that the higher the interest rate charged by the bank, i_b, the less the percentage of the bank loan. Below is an example that illustrates the importance of the expressions in equations (8.5) and (8.6) in determining the optimal amount of loan for a business.

EXAMPLE 48. *Suppose you calculated the initial investment to be* $I = \$100,000$. *You are planning to raise this capital through debt and equity financing. The question that you are trying to answer is: What is the optimal capital structure? In other words, how much of the $100,000 should be debt financing and how much should be equity financing? Suppose that the average bank rate on loans is 2%; that is,* $i_b = 2\%$. *Suppose that you are planning to repay the loan over 5 installments; that is,* $s = 5$. *This is reasonable since the expected life time of a typical simple project is 5 years. Now, consider equation (8.5): Given* i_b, I, s, *and* r, *equation (8.5) gives the optimal repayment of loan,* R, *that guarantees profit maximization for the investor/borrower and liquidity of the lender. We already have* i_b, I, *and* s. *We need to find the rate of return on the project,* r. *Remember, however, that we are performing a feasibility study; that is, the project doesn't actually exist. Thus, we don't have an actual rate of return for it since we didn't start operations yet. We are in the planning phase, where we are assessing the viability of this project and, based on the assessment, we may or may not invest in the project. In this case, we use a target rate of return as a proxy for* r. *This is the rate of return that you, as an investor, would expect to achieve on this type of project. This rate could be taken to be the average industry return or the expected rate of return on a similar project in the same risk class. Suppose the target rate of return is* $r = 8\%$. *Now, equation (8.5) yields and optimal* R *as*

$$\begin{aligned} R &= \frac{I \times r}{1 + (s \times i_b)} \\ &= \frac{100000 \times 0.08}{1 + (5 \times 0.02)} \\ &= \$7272.7; \end{aligned}$$

that is, the optimal repayment (per installment) is approximately $7273. It follows then, from equation (8.6), that the optimal amount of loan is $36,365 as

$$L = s \times R = 5 \times 7273 = \$36,365.$$

Thus, the optimal amount of capital that should be raised via debt is $36,365. By "optimal" here, we mean the amount of loan that will guarantee profit maximization and liquidity. It follows then that the optimal amount of capital that should be raised via equity is

$$100000 - 36365 = 63635.$$

In other words, the firm's optimal capital structure is such that the weight of debt is 0.36 and the weight of equity is 0.64.

REMARK 15. *The debt to equity ratio will depend up on the relation between the return on investment r and the interest rate i_b. The higher the difference between the two, the higher will be the percentage of the loan.*

6. Step 4: Financial Evaluation

Once the optimum financial structure is identified, you have all the information to evaluate the project. Before we proceed, it is worth noting the difference between **accounting profit** and **financial profit.** Recall the conventional pro-forma income statement of any business:

	Revenue
-	Cost of operations
	GOP (Gross Operating Profit)
-	Depreciation
	NPBI (net profit before interest)
-	interest
	NPBT (net profit before tax)
-	Taxes
	NPAT (net profit to shareholders)

The net profit to shareholders is called "accounting profit." Notice that depreciation and interest expenses are deducted when calculating the accounting profit. The reason is that accounting profit is concerned with the profit to shareholders. Since interest is an obligation on the shareholders, it has to be deducted from the profit. Depreciation is the proportion of the assets that is worn out during the expected life time of the asset. It is also deducted from profit for accounting reasons.

In finance, we evaluate an investment not an equity. In finance, interest is not an obligation and thus, we do not deduct interest expenses. Also, we do not account for depreciation because we are concerned with the actual cash in due time. Therefore, in financial evaluation, we have to convert accounting profit into financial profit as

$$GOP - Taxes - change\ in\ investment = Net\ Cash\ Flow\ (NCF).$$

In financial evaluation we deduct taxes and any investment during that year from the gross operating profit to arrive at the net cash flow, NCF, or the **financial profit**. Note that net profit, i.e., profit to shareholders, is the accounting profit whereas NCF (profit to the project) is the financial profit.

6.1. How can we use the NCF to evaluate a Project? We have two methods to evaluate a project: (a) simple methods and (b) discounted methods. The simple methods do not consider the entire life of the project. The annual data used are taken based on the actual data and not on the discounted date, i.e., these methods ignore the value of money over time. The payback period (PBP) is an example of a simple method of financial evaluation.

6.1.1. *(a) Payback Period.* Recall, from Chapter 7, that the payback period is the period required to compensate the original investment cost through the profits earned by the project. Consider the following example.

EXAMPLE 49. *Suppose the total investment cost of a given project is \$10300 and the projected net cash flows of this project are as follows*

Year	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7
NCF	construction period	construction period	870	2030	2330	2500	3500

The payback period can be calculated as follows:

	Amount Paid Back	Balance at the End of the Year
Year 1	–	10300
Year 2	–	10300
Year 3	870	9430
Year 4	2030	7400
Year 5	2330	5070
Year 6	3500	1570
Year 7	3500	

The calculations indicate that the original investment cost will be recovered after nearly 6.5 years. The major drawback of the payback period is that it ignores the value of money over time.

6.1.2. *(b) Discounted methods of financial evaluation.* The rationale here is to consider the discounted earnings of the project, i.e., to project the future cash flows that are expected to be generated by the project and discount them using the opportunity cost of the capital invested in the project. The opportunity cost of the capital invested depends on the source of capital used. We have two main sources of capital used to finance a given project: (a) equity and (b) loans. The opportunity cost of capital is the weighted average of the cost of both sources and is called the weighted average cost of capital (WACC).

6.2. How to estimate the WACC in practice? The WACC is estimated using the following formula

$$WACC = \left(\left(i_d + \underbrace{(\gamma \times \beta)}_{RP=\text{ risk premium}}\right)\left(\frac{E}{I}\right)\right) + \left(i_b(1-t)\left(\frac{L}{I}\right)\right), \tag{8.7}$$

where:

$E =$ part of capital financed through equity,

$L =$ part of capital finance through loans,

$I =$ investment cost,

$i_d =$ interest rate on deposits,

$i_b =$ interest rate on borrowings (loans),

$t =$ tax rates,

$\gamma =$ risk coefficient, and

$\beta =$ the beta coefficient (measure of risk).

The cost of borrowings through loans is the interest rate on borrowings i_b. Loan financing is more appealing to the entrepreneurs as it reduces the cost of finance by the tax rate. To illustrate this idea, consider the following example.

EXAMPLE 50. *Consider two projects: A and B, where project A is subject to a loan that costs 20 as interest expense. Both projects are subject to a tax rate of 30%.*

Project	A	B
NPBI	100	100
Interest	20	-
Net Earnings	80	100
Taxes (30%)	24	30

Observe how project A pays more interest expenses and less taxes and, therefore, saves $30 - 24 = 6$. In other words, the interest charges are reduced by the tax rate. The cost of loan financing of the firm is not 20, it is rather 14 $[i_b(1-t) = 20(1-0.3) = 14]$. Therefore,

$$\textit{Opportunity cost of borrowing} = i_b(1-t). \tag{8.8}$$

The opportunity cost of equity is $i_d + RP$, where i_d is the interest on deposit and RP is the risk premium. The problem is how to evaluate this risk. We usually estimate the country risk (risk of investing in a specific country). This involves estimating some economic, political, financial, and social parameters. Focusing on economic and financial parameters or, in other words, assuming a neutral political and social factors, there are 5 basic parameters to consider:

(1) The rate of growth of GDP (g); this is a positive factor.
(2) The inflation rate of the country (π); this is a negative factor.
(3) Depreciation of the value of the local currency; this is a positive factor if it is a foreign direct investment.
(4) Unemployment rate (U); this is a negative factor.
(5) Percentage of businesses withdrawn; this is a negative factor.

We give weights to each factor and come up with a number, γ, reflecting the country's risk. Once we estimate the country risk, we adjust it by including the risk of the sector. This is done by using the beta coefficient, β, which is a reflection of the risk of a sector as compared to the average risk of a country. The decision criterion is:

$$\begin{aligned}
\beta &> 1 \Longrightarrow \textit{Sector risk} > \textit{country risk};\\
\beta &= 1 \Longrightarrow \textit{Sector risk is within the country risk};\\
\beta &< 1 \Longrightarrow \textit{Sector risk} < \textit{country risk}.
\end{aligned}$$

If the industry relies on certain inputs that has to be imported, beta would be higher than the average.

$$\textit{Opportunity cost of Equity} = i_d(\gamma \times \beta) \tag{8.9}$$

From (8.8) and (8.9), the WACC in (8.7) is calculated as

$$WACC = \left(i_d + (\gamma \times \beta)\left(\frac{E}{I}\right)\right) + \left(i_b(1-t)\left(\frac{L}{I}\right)\right),$$

where $\left(\frac{E}{I}\right)$ and $\left(\frac{L}{I}\right)$ are the weights and they sum to 1; that is

$$\frac{E}{I} + \frac{L}{I} = 1.$$

The WACC is not the same for all projects. It is affected by four factors: (1) the equity/loan ratio, (2) whether the firm is subject to taxes or not, (2) the beta coefficient, which is a reflection of the sector risk, and (4) the risk coefficient γ.

REMARK 16. *The computed WACC is the nominal WACC. The real WACC is obtained by subtracting the inflation rate from the nominal WACC.*

6.3. The Discounted Cash Flow Method.

(8.10)

$$PV = \frac{NCF_1}{(1+WACC)} + \frac{NCF_2}{(1+WACC)^2} + ... + \frac{NCF_n}{(1+WACC)^n} + \frac{\text{Residual}}{(1+WACC)^{n+1}},$$

where:

NCF = is the net cash flow over the expected life time, n, of the project.

$WACC$ = is the weighted average cost of capital, and

PV = is the present value of the project as a going concern.

REMARK 17. *If the NCF is based on constant prices of the base year, you should deduct the inflation rate from the WACC. If the NCF is based on nominal prices, keep the WACC at current prices.*

The decision criterion that is used to determine the viability of the project is based on comparing the PV of the project (value of the project as a going concern) and the investment cost (the market value of the project's assets). The difference between the PV of the project and the investment cost (I) is called the net present value (NPV).

$$NPV = PV - I$$

The decision criterion is

$$\begin{aligned} \text{If } PV &> I \Longrightarrow NPV > 0 \Longrightarrow \textit{The project is viable} \\ \text{If } PV &= I \Longrightarrow NPV = 0 \Longrightarrow \textit{border line.} \\ \text{If } PV &< I \Longrightarrow NPV < 0 \Longrightarrow \textit{The project is not viable} \end{aligned}$$

6.4. The Internal Rate of Return (IRR). The question is to what extent the project is viable? To answer this question, we need to determine the discount rate that equates today's value, i.e., the investment cost, to the present value of all expected net cash flows from the project. In other words, we need to find the yield of this project. This rate, as you know by now, is called the internal rate of return (IRR). The IRR solves the following equation

$$(8.11)\ I = \frac{NCF_1}{(1+IRR)} + \frac{NCF_2}{(1+IRR)^2} + ... + \frac{NCF_n}{(1+IRR)^n} + \frac{\text{Residual}}{(1+IRR)^{n+1}}.$$

$$\begin{aligned} \text{If } PV &> I \Longrightarrow NPV > 0 \Longrightarrow IRR > WACC; \\ \text{If } PV &= I \Longrightarrow NPV = 0 \Longrightarrow IRR = WACC; \\ \text{If } PV &< I \Longrightarrow NPV < 0 \Longrightarrow IRR < WACC. \end{aligned}$$

7. Step 5: Sensitivity Analysis

When you evaluate a project, generally you want to estimate the break-even point (BEP); the point at which the sales revenue is equal to production costs, i.e., total revenue equals to total cost ($TR = TC$). The BEP can also be defined in terms of the physical units produced or the level of capacity utilization at which $TR = TC$. The break-even analysis is used to assess the riskiness of a project. If the BEP is reached at a higher level of output, the project is considered risky and losses are expected as a result of any decline in productivity. If, on the other hand, the BEP is reached at a lower level of output, the project is a sound project and there will be still room for profit in case of a decrease in productivity.

Let P be the product price, Q be the quantity produced, TFC be the total fixed cost, and TVC be the total variable cost. Define the average variable cost, AVC, by the equation

$$AVC = \frac{TVC}{Q},$$

total revenue, TR, by the equation

$$TR = P \times Q,$$

and total cost, TC, by the equation

$$TC = TFC + TVC.$$

Given the previous notions, the formula to estimate the BEP can be derived as follows. At break-even,

$$\begin{array}{rcl} TR & = & TC\text{; that is,} \\ Q \times P & = & TFC + TVC\text{, or} \\ Q \times P & = & TFC + (Q \times AVC) \\ & \Longrightarrow & Q(P - AVC) = TFC, \end{array}$$

or simply,

$$Q_{BE} = \frac{TFC}{P - AVC}, \tag{8.12}$$

where $(P - AVC)$ is the profit margin per unit.

REMARK 18. *Sometimes you have more than one product and, thus, have different prices. In such a case, the Q_{BE} is defined as the percentage of capacity of your project (percentage of your product mix) as*

$$Q\% = \frac{TFC}{TR - TVC}. \tag{8.13}$$

Sensitivity analysis is after developing scenarios to assess how the profitability of a project changes when we alter the values of some parameters, e.g., per unit price, per unit cost, sales volume and so on. Sensitivity analysis is used when the evaluator thinks that changing some of the parameters will lead to an improvement in profitability.

Sensitivity analysis should be applied during the project planning stage, when decisions concerning major inputs are being taken. So, different scenarios, optimistic and pessimistic ones, are developed to asses the effect on the project. Note that by selecting the pessimistic scenario, one can determine the viability of the project is the worst possible case.

The application of sensitivity analysis can be illustrated by showing the impact of changing sales price per unit and variable production and fixed production costs on the BEP.

EXAMPLE 51. *Assume that the projected sales revenue for project A are \$12500; the fixed production costs are \$3280 and the variable production costs are \$6500. Assume also that the projected number of units that will be produced is 2000 units and the break-even is calculated for capacity utilization. Suppose the unit price first changes from \$6.25 to \$5.75 and then to \$5.50. Analyze this pessimistic scenario using sensitivity analysis.*

Recall the break-even point formula as a percentage of capacity

$$Q_{BE}(\%\ of\ capacity) = \frac{Fixed\ production\ costs}{Sales\ revnue - variable\ production\ costs}. \tag{8.14}$$

Using our data, the BEP would be reached at a production output of

$$Q_{BE} = \frac{TFC}{P - AVC} = \frac{3280}{6.25 - 3.25} = 1093, \tag{8.15}$$

where

$$AVC = \frac{TVC}{Q} = \frac{6500}{2000} = 3.25.$$

Therefore, to break-even, the quantity produced should be 1093 units. This can be expressed in terms of capacity utilization using the formula in (8.14) as

$$BEP(\%) = \frac{3280}{(2000 \times 6.25) - 6500} \times 100\% = 55\%, \tag{8.16}$$

where

$$Sales\ revnue = P \times Q = 6.25 \times 2000 = 12500$$

Now, consider a drop in price from $P = \$6.25$ to $P_1 = \$5.75$ and then to $P_2 = \$5.25$. First, the sales volume will be affected and, consequently, the break-even point as a percentage of capacity will change. The new sales volume will be

$$Sales\ revnue_1 = P_1 \times Q = 5.75 \times 2000 = 11500$$

and the new break-even point will be

$$BEP_1 = \frac{3280}{11500 - 6500} \times 10\% = 65\%.$$

The 65% means 1300 units, which is equivalent of \$7475 sales revenue. A further drop in price by 25 cents will lead to a new break-even point

$$BEP_1 = \frac{3280}{11000 - 6500} \times 10\% = 73\%.$$

The 73% means 1460 units, which is equivalent of \$8030 sales revenue.

8. Practice Problems

Find an interesting project that you would like to invest in and perform a simple financial evaluation using the tools that you learned in MGMT560. Below are the points that you need to cover in your study

- **Market Information**
  - Identify the type of market structure that you will be operating in.
  - Estimate your market segment in the industry and your market share. (Hint: you can use the concentration ratio to guesstimate your market share; it is simply the ratio of your sales relative to the industry's sales).
  - Estimate the market price of your final product(s) and your inputs (wether imported or purchased domestically).
  - Estimate any other market information that is relevant to your project.
  - Identify the life time of your project and estimate the residual value in case of liquidation.
- **Technical Information**
  - Information relevant to the premises and location, i.e., size, width, length, height, renovated or not, and so on.
  - List the machines and equipment that you will potentially use and guesstimate the cost of each item.
  - Estimate the costs of operations: direct costs and indirect costs.
  - Estimate your variable costs.
  - Estimate the capacity of your project and the number of units that you will potentially sell.
- **Financial Evaluation**
  - Compute your yearly revenue and project/forecast an adequate rate of growth over the life time of the project. Do the same growth projection exercise for your costs.

  - After breaking down the revenues and costs, prepare a yearly proforma income statement over the life time of the project and compute the annual gross operating profit. Did you find sensible results? Why or why not?
  - Estimate the fixed investment and the net working capital (NWC) over the life time of the project. Show your work.
  - Compute the net cash flows (NCFs) over the expected life time of your project.
  - Let the country risk be 2% and the beta coefficient in your sector be 0.5. Estimate the weighted average cost of capital. Show your work.
  - Bonus: Estimate the beta coefficient in your sector rather than using 0.5.
- **Discounting the Cash Flows**
  - Use all previous data to assess the viability of your project using the payback period method and the net present value method. Provide conclusions.
  - Find the IRR on your project and compare it to the WACC.
- **Sensitivity Analysis**
  - Perform a sensitivity analysis to your results. In particular, consider two scenarios: (1) an optimistic scenario, where you anticipate a 25% surge in the price of your final product and (2) a pessimistic scenario, where the price drops by 25%. Comment on your results.

CHAPTER 9

Capital Structure and Dividend Policy

1. Introduction: Financial Leverage

The issue of deciding on the optimal **capital structure** of a firm is a key topic in corporate finance. Unfortunately, the process of determining the optimal capital structure of a particular firm is not unique; every financial manager has to devise his or her own strategy to determine the optimal capital mix. On the research frontier, scholarly articles suggest that there is an optimal range of capital structure. Despite the fact that these models do not specify the optimal mix of capital, they still provide significant guidance in understanding how the choice of the firm's finance mix affects its value.

As for **dividend policy**, since the expected cash dividends are the main ingredients that shareholders use to determine the share value, then the decision to allocate the net profit after taxes between retained earnings and dividends is worth exploring for it can affect the firm's external financing requirements. The larger the cash paid out as dividends, the more the firm needs financing.

It is important to understand one main distinction between capital structure decisions and dividend policy decisions. Capital structure policy can be thought of as a way of creating value to the firm whereas dividend policy can be thought of as a way of delivering this value. It turned out, as we will see shortly, that, in a world with no taxes, it makes no difference how this value is delivered, i.e., it makes no difference whether the shareholders receive their cash flows as dividends or capital gains. This is the presumption of dividend irrelevance theory,–in the absence of taxes and other distortions, dividend policy is irrelevant. However, once we consider personal and corporate taxes, the possibility that dividends may affect value must be considered.

It is assumed that the reader brings the background of an introductory course in accounting. Exposure to basic notions such as financial statements, sources and uses of funds, financial ratios, and financial leverage is expected. Here, at the outset, we briefly discuss a key notion in the upcoming analysis,–**financial leverage**.

A quick inspection of the right-hand-side of a consolidated balance sheet reveals that any firm can raise funds through borrowing from a third party.

This third party could be a financial intermediary, e.g., a financial bank. In this case, the borrowing is in the form of a short-term or long-term loan. Recall that this type of financing is known as **debt financing** or fixed-cost financing since the cost of financing, which is the interest rate on the loan, is fixed. Another type of debt financing, as discussed in Chapter 1, is raising funds by issuing a corporate bond. Alternatively, the firm could raise funds by issuing common shares (CS) or preferred shares (PS). Borrowing via issuing CS or PS is considered **equity financing**. The key difference between equity financing and debt financing lies in the finance cost. Bond holders are entitled to receive a **fixed** amount known as bond interest every specific period of time till maturity. On the other hand, CS holders are entitled to receive dividends according to the firm's dividend policy. Dividend payments are naturally variable, and are allocated from the firm's net income at the end of the fiscal year as per the firm's dividend policy. The firm is obliged to pay its debts first, and for that reason, debt holders are sometimes called **first claimants** whereas shareholders are **residual claimants**.

A key concept in the discussion of capital structure is **financial leverage**. In general, there are two different types of leverage that any firm could be exposed to; **operating leverage** and **financial leverage**. The former can be defined as the potential use of **fixed operating costs** to magnify the effects of changes in sales on the firm's earning before interest and taxes (EBIT). The latter results from the use of **fixed-cost** assets or funds to magnify returns to the firm's owners. As you probably noticed, the idea of leverage is tied to the use of fixed costs in financing operations or capital. Since our focus is on capital structure, we will discuss the latter.

DEFINITION 55. *Financial leverage is the magnification of risk and return through fixed asset financing. The amount of leverage in the firm's capital structure is the relative amount of fixed cost of capital, i.e., cost of debt and preferred stocks, relative to the total cost of capital. This amount can significantly affect the firm's value through its direct effect on the firm's risk and return.*

Financial leverage, by definition, creates financial risk. To see how financial leverage can affect risk and return, consider the following simple example of two hypothetical firms: a levered firm, L, and an unlevered firm, U. Assume further that both firms are exactly identical except for the way they are financed; firm U finances 100% of its total assets through equity whereas firm L finances 50% of its total assets through debt at a cost of 25% and the remaining 50% through equity as shown from the consolidated balance sheets of the two firms in the tables below respectively. Both firms have \$100 of assets and \$100 of sales. A quick look at the consolidated income statements of both firms, one can notice that both firms generate the same EBIT. Firm L is, however, better under normal conditions for it has a higher return on equity (ROE) of 27% as opposed to that generated by firm U, which is only 18%. This is because firm L uses debt to finance its operations, and therefore, is subject to what is known as the "**leverage effect**," which means increasing the expected rate of return to shareholders, as shown from the ROE, but also increasing risk under bad conditions, i.e., when sales decrease or costs increase or both. In this example, bad conditions are represented by a decrease in sales as shown from the consolidated income statements in the tables below. Notice that under bad conditions, the EBIT will decrease significantly and the payment of interest on debt will, consequently, reduce net income, which, in turn, could result in losses, i.e., negative ROE under bad conditions.

Firm L:	**Consolidated Balance Sheet**			
	Current Assets	50	Debt	50
	Fixed Assets	50	Common Equity	50
	Total Assets	*100*	***Total Liability & Equity***	*100*
	Expected Consolidated Income Statement			
	Conditions:	**Good**		**Bad**
	Sales	100		82.5
less	Operating Costs	70		80
	EBIT	30		2.5
less	Interest	7.5		7.5
	EBT	22.5		(5)
less	Taxes (40%)	9		2
	Net Income	13.5		(3)
	ROE	27%		6%

Firm U:	**Consolidated Balance Sheet**			
	Current Assets	50	Debt	0
	Fixed Assets	50	Common Equity	100
	Total Assets	100	**Total Liability & Equity**	100
	Expected Consolidated Income Statement			
	Conditions:	**Good**		**Bad**
	Sales	100		82.5
less	Operating Costs	70		80
	EBIT	30		2.5
less	Interest	0		0
	EBT	30		2.5
less	Taxes (40%)	12		1
	Net Income	18		1.5
	ROE	18%		1.5%

2. The Theory of Capital Structure

The foundation of capital structure theory is due to the work of Modigliani and Miller (1958, 1963), who showed that under the assumption of perfect capital markets, the capital structure chosen by the firm does not affect its value. The relationship between capital structure and the firm's value is the key to the theory of capital structure. Since Modigliani and Miller's (1958, 1963) work, the relation has been examined further by many researchers under less restrictive assumptions. The result is a theoretical optimal capital structure based on balancing the benefits and costs of **debt financing**. The benefit of debt financing is its leverage effect, i.e., the tax shield provided, which allows interest payments to be deducted in calculating taxable income. This yields less taxes and higher ROE. The costs of debt financing are due to the financial risks associated with it. Copeland et al (2005) summarize the costs of debt financing in three main costs:

(1) The increased probability of bankruptcy due to debt obligations.
(2) The agency cost of the lender's monitoring of the firm's actions.
(3) The costs associated with managers having more information about the firm's prospects that do investors.

We begin with the seminal paper of Modigliani and Miller (1958) or M-M model for short, which is after studying the effect of tax deductible debt on the value of the firm. To simplify the analysis, we will first study the relation in a world with only corporate taxes, then we will make the model more realistic by adding personal taxes.

2.1. The Value of the Firm Given Corporate Taxes Only (M-M model).

Modigliani and Miller set forth the following axioms to their model:

AXIOM 1. *Capital markets are efficient.*

AXIOM 2. *Investors can borrow and lend at the risk-free rate.*

AXIOM 3. *There are no costs to bankruptcy or business disruption, i.e., financing through debt is risk-free.*

AXIOM 4. *Firms issue only two types of claims: risk-free debt and risky equity.*

AXIOM 5. *Operating risk is the same for all firms, i.e., all firms are assumed to be in the same risk class.*

AXIOM 6. *Corporate taxes are the only form of government taxes, i.e., no personal taxes or wealth taxes for instance.*

AXIOM 7. *All cash flow streams are perpetuities, i.e., no growth is assumed.*

AXIOM 8. *Corporate insiders and outsiders have the same information, i.e., no signaling opportunities.*

AXIOM 9. *Managers are always after maximizing shareholders' wealth, i.e., no agency costs.*

AXIOM 10. *Operating cash flows are completely unaffected by changes in capital structure.*

Although some of the previous assumptions are restrictive and might be unrealistic, relaxing them does not really affect the main conclusion of the M-M model. The only exceptions are Axiom 3 and Axiom 5. Relaxing these assumptions has serious implications on the model. For instance, relaxing Axiom 3 can seriously affect the value of the firm, and consequently, the main conclusion of the model. Axiom 5, on the other hand, means that all firms have same operating risk. This implies that the expected risky future net operating cash flows vary by, at most, a scale factor; that is, for any two firms or projects i and j,

$$\widetilde{CF}_i = \lambda \widetilde{CF}_j,$$

where $\widetilde{CF}$ is the risky net cash flow from operations, i.e., cash flow before interest and taxes and λ is a constant scale factor. This implies that the expected future cash flows from any two firms/projects i and j are perfectly correlated. This perfect correlation is manifested in the firms' returns as follows: Let $\widetilde{r}_{it}$ be the rate of return on firm/project i at time t be defined by the usual percentage change formula:

$$\widetilde{r}_{it} = \frac{\widetilde{CF}_{it} - \widetilde{CF}_{it-1}}{\widetilde{CF}_{it-1}}.$$

If $\widetilde{CF}_i = \lambda \widetilde{CF}_j$, then

$$\widetilde{r}_{it} = \frac{\lambda \widetilde{CF}_{jt} - \lambda \widetilde{CF}_{jt-1}}{\lambda \widetilde{CF}_{jt-1}} = \widetilde{r}_{jt}.$$

Therefore, Axiom 5 implies that if two streams of cash flows differ by, at most, a scale factor, they will have the same distribution of returns, the same risk, and will require the same expected return. Relaxing this assumption will change the conclusion of the model. At any rate, the conclusion of M-M model is valid for firms/projects operating in the same risk class.

Now, we are ready to state and prove the main proposition of M-M model. In the literature on the subject, the following proposition is known as **Modigliani-Miller Proposition 1**.

PROPOSITION 1. ***Modigliani-Miller Proposition 1***. *The market value of any firm is independent its capital structure and is given by capitalizing its expected return at a rate, ρ, that is appropriate to its class, provided* *that Axioms 1-10 are satisfied. In other words the method of financing is irrelevant.*

In order to prove the previous proposition, we need to show that the market value of a levered firm is the same as that of an unlevered firm if corporate tax rate is zero; that is, we need to show that

$$V_L = V_U \text{ if } \tau_c = 0, \tag{9.1}$$

where V_L is the present value of an levered firm, i.e., a firm financed by debt and equity, V_U is the present value of an unlevered firm, i.e., a firm financed only by equity, and τ_c is the corporate tax rate.

Before we proceed with the proof of the previous proposition, recall that a pro-forma income statement for a typical firm takes the following form:

Revenues
$-$ Variable Costs of Operations (VC)
$-$ Fixed Cash Costs (FCC), e.g., administrative and real estate taxes)
$-$ Depreciation (DEP)
$=$ $EBIT$
$-$ $k_d D$ (interest on debt)
$=$ EBT
$-$ T (taxes: $T = \tau_c \times EBT$)
$=$ NI

Notice that the total fixed costs are partitioned in two parts: FCC and DEP. The former is the cash part and the latter is the non-cash part as shown from the pro-forma income statement.

For an **unlevered firm**, i.e., when debt financing is zero, i.e., $D = 0$, after-tax cash flows from operations are calculated as: After-tax operating income plus depreciation and any other non cash expenses; that is,

$$\begin{aligned}
\text{After-tax Cash Flow} &= \text{After-tax Operating Income} + \text{DEP} \\
&= \underbrace{\left(\widetilde{REV} - \widetilde{VC} - FCC - DEP\right)}_{=\widetilde{EBIT}} - T + DEP \\
&= \widetilde{EBIT} - \tau_c \widetilde{EBIT} + DEP \\
&= \widetilde{EBIT}\,(1 - \tau_c) + DEP,
\end{aligned}$$

or equivalently,

$$\text{After-tax Cash Flow for Unlevered Firm } = \widetilde{EBIT}\,(1 - \tau_c) + DEP.$$

Note that if a no growth assumption is entertained, i.e., if all cash flows are perpetuities, the depreciation each year must be replaced by investment, I, in order to keep the same amount of capital in place. This implies that the after-tax free cash flow for an unlevered firm, FCF_U, that is available for payment to creditors and shareholders is

$$\begin{aligned}\widetilde{FCF_U} &= \left(\widetilde{REV} - \widetilde{VC} - FCC - DEP\right)(1-\tau_c) + DEP - I \\ \text{(9.2)} \qquad &= \underbrace{\left(\widetilde{REV} - \widetilde{VC} - FCC - DEP\right)}_{=\widetilde{EBIT}}(1-\tau_c) \text{ (since } DEP = I).\end{aligned}$$

On the other hand, for a **levered firm**, i.e., when debt is not zero, i.e., $D \neq 0$, after-tax cash flows must be split between debt holders an shareholders. The latter receives $NI + DEP - I$ whereas the former receives $k_d D$. Thus,

$$\begin{aligned}\widetilde{FCF_L} &= \widetilde{NI} + DEP - I + k_d D \\ &= \widetilde{NI} + k_d D \\ &= \left(\widetilde{REV} - \widetilde{VC} - FCC - DEP - k_d D\right)(1-\tau_c) + k_d D \\ \text{(9.3)} \qquad &= \underbrace{\left(\widetilde{REV} - \widetilde{VC} - FCC - DEP\right)}_{=\widetilde{EBIT}}(1-\tau_c) + k_d D \tau_c.\end{aligned}$$

Now, let ρ be the discount rate for an all equity firm, which in practice is usually taken to be the WACC, k_b be the before-tax market required rate of return on a risk-free stream of perpetual bond, $E\left[\widetilde{FCF_L}\right]$ is the expected value of perpetual free cash flow after taxes for a levered firm, and $E\left[\widetilde{FCF_U}\right]$ is the expected value of perpetual free cash flow after taxes for an unlevered firm. Then, the value of an unlevered firm is

$$\begin{aligned}V_U &= \frac{E\left[\widetilde{FCF_U}\right]}{1+\rho} + \frac{E\left[\widetilde{FCF_U}\right]}{(1+\rho)^2} + \frac{E\left[\widetilde{FCF_U}\right]}{(1+\rho)^3} + \cdots \\ &= \frac{E\left[\widetilde{FCF_U}\right]}{1+\rho}\underbrace{\left(1 + \frac{1}{1+\rho} + \left(\frac{1}{1+\rho}\right)^2 + \left(\frac{1}{1+\rho}\right)^3 + \cdots\right)}_{=\frac{1}{1-\frac{1}{1+\rho}}=\frac{1+\rho}{\rho}} \\ &= \frac{E\left[\widetilde{FCF_U}\right]}{\rho},\end{aligned}$$

or more explicitly using equation (9.2),

$$\begin{aligned}V_U &= \frac{E\left[\widetilde{FCF_U}\right]}{\rho} = \frac{E\left[\left(\widetilde{REV} - \widetilde{VC} - FCC - DEP\right)(1-\tau_c)\right]}{\rho} \\ \text{(9.4)} \qquad &= \frac{E\left[\widetilde{EBIT}(1-\tau_c)\right]}{\rho},\end{aligned}$$

and the value of a levered firm is

$$
\begin{aligned}
V_L(9.5) \quad & PV\left(\widetilde{FCF}_L \text{ to shareholders}\right) + PV\left(\widetilde{FCF}_L \text{ to bondholders}\right) \\
= \quad & \frac{E\left[\widetilde{FCF}_U\right]}{1+\rho} + \frac{E\left[\widetilde{FCF}_U\right]}{(1+\rho)^2} + \cdots + \frac{k_d D \tau_c}{1+k_b} + \frac{k_d D \tau_c}{(1+k_b)^2} + \cdots \\
= \quad & \frac{E\left[\widetilde{EBIT}(1-\tau_c)\right]}{\rho} + \tau_c \left\{ \frac{k_d D}{1+k_b} \underbrace{\left(1 + \frac{1}{1+k_b} + \left(\frac{1}{1+k_b}\right)^2 + \cdots\right)}_{=\frac{1+k_b}{k_b}} \right\} \\
= \quad & \frac{E\left[\widetilde{EBIT}(1-\tau_c)\right]}{\rho} + \tau_c \frac{k_d D}{k_b},
\end{aligned}
$$

where $\frac{E\left[\widetilde{EBIT}(1-\tau_c)\right]}{\rho} = V_U$ and $\frac{k_d D}{k_b}$ is simply the market present value of the bond. Let $B = \frac{k_d D}{k_b}$, then we can express (9.5) as

$$V_L = V_U + \tau_c B. \tag{9.6}$$

Equation (9.6) states that the value of the levered firm, V_L, is equal to the value of the unlevered, V_U, firm plus the present value of the tax shield provided by debt, $\tau_c B$. This extra value created by the interest tax shield on debt is known as the gain from leverage. It is obvious that if $\tau_c = 0$, then equation (9.6) becomes

$$V_L = V_U \qquad (\text{if } \tau_c = 0). \tag{9.7}$$

Equation (9.7) states that in the absence of any market imperfections including corporate taxes, i.e., if $\tau_c = 0$ and all other imperfections do not exist, the value of the firm is completely independent of the type of financing using for its projects. Therefore, according to Modigliani and Miller, the market value of any firm is independent of its capital structure and is given by the ratio of its expected EBIT to a rate ρ appropriate to its risk class. Hence, the method of financing is irrelevant.[1]

2.2. Adding Personal Taxes. In the previous sub-section, we have seen that in a perfect world without taxers, the market values of levered and unlevered firms are identical. In this sub-section we analyze the value of the firm in a world with both personal and corporate taxes. This analysis is due

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[1] If the no growth assumption is entertained and if $\tau_c = 0$, then

$$V_L = V_U = \frac{E\left[\widetilde{EBIT}\right]}{\rho},$$

where ρ is taken to be the WACC.

to Miller (1977), who modified the result in (9.6) by introducing personal taxes as well as corporate taxes in the model.

Following Miller's (1977) model, we will consider two types of personal taxes, namely, taxes on income received from holding shares and taxes on income from bonds. Let τ_{PS} denote the personal tax rate on income received from holding shares and τ_{PB} denote the personal tax rate on income from bonds.

PROPOSITION 2. *(Miller (1977)). Suppose that Axioms 1-10 of M-M hold. Then, in a world with both personal and corporate taxes,*

$$V_L = V_U + \left[1 - \frac{(1-\tau_c)(1-\tau_{PS})}{(1-\tau_{PB})}\right] B, \tag{9.8}$$

where the term in brackets in the right-hand-side times the market value of debt, denoted by B, is the gain from financial leverage in this case.

To prove the previous proposition, notice that, given the existence of personal income taxes, namely, given τ_{PS} and τ_{PB}, the value of an unlevered firm is, therefore,

$$V_U = \frac{E\left[\widetilde{EBIT}(1-\tau_c)(1-\tau_{PS})\right]}{\rho}.$$

On the other hand, if the firm has both bonds and shares outstanding, the earning stream is partitioned into 2 parts: (a) cash flows to shareholders after corporate and personal taxes, which are equal to

$$\left(\widetilde{EBIT} - k_d D\right)(1-\tau_c)(1-\tau_{PS}), \tag{9.9}$$

and (b) payments to bond holders after personal taxes, which are equal to

$$k_d D(1-\tau_{PB}). \tag{9.10}$$

Therefore, the total cash payments to suppliers of capital is the sum of (9.9) and (9.10); that is,

$$\begin{aligned} \text{Total Cash Payments} &= \left(\widetilde{EBIT} - k_d D\right)(1-\tau_c)(1-\tau_{PS}) + k_d D(1-\tau_{PB}) \\ &= \widetilde{EBIT}(1-\tau_c)(1-\tau_{PS}) \\ &\quad -k_d D(1-\tau_c)(1-\tau_{PS}) \\ &\quad +k_d D(1-\tau_{PB}). \end{aligned} \tag{9.11}$$

The first term of equation (9.11) is the same as the cash flows to owners of unlevered firms, and thus, should be discounted at the cost of equity for an all-equity firm, which is ρ. The second and the third terms should be discounted at the risk-free rate, k_b. Hence,

$$V_L = \frac{E\left[\widetilde{EBIT}(1-\tau_c)(1-\tau_{PS})\right]}{\rho} + \frac{k_d D\left[(1-\tau_{PB}) - (1-\tau_c)(1-\tau_{PS})\right]}{k_b}, \tag{9.12}$$

or equivalently,

$$V_L = V_U + \left[1 - \frac{(1-\tau_c)(1-\tau_{PS})}{(1-\tau_{PB})}\right] B, \tag{9.13}$$

where

$$B = \frac{k_d D (1-\tau_{PB})}{k_d} \tag{9.14}$$

is the market value of debt.

Notice that the gain from leverage in a world of corporate and personal taxes is

$$\left[1 - \frac{(1-\tau_c)(1-\tau_{PS})}{(1-\tau_{PB})}\right] B.$$

Notice also that when personal taxes are set to zero, equation (9.13) is reduced to equation (9.6); the result of M-M model. The second important thing to note is that if

$$1 = \frac{(1-\tau_c)(1-\tau_{PS})}{(1-\tau_{PB})} \iff (1-\tau_{PB}) = (1-\tau_c)(1-\tau_{PS}),$$

the advantage of debt financing may be lost. In other words, if the personal income tax on stocks is less than the tax on income from bonds, i.e., if $\tau_{PS} < \tau_{PB}$, then the gain from leverage is lower, and therefore, from equation (9.13), the market value of the bond has to increase to offset this advantage.

3. Dividend Policy

3.1. Arguments for Dividend Irrelevance. The **residual theory of dividends** has its roots in the work of Preinreich (1932) and Sage (1937), who were probably the first to provide an academic description of what it means by residual dividends. This school of thought suggests that the dividend paid by a firm should be viewed as a **residual**,—an amount left over after all acceptable investment opportunities have been considered. In general the residual theory of dividends suggests that the firm should treat the dividend decision in three steps: First, it should determine the optimal level of capital expenditures. Second, using the optimal capital structure proportions, the firm can estimate the total equity needed to meet the optimal capital expenditures. Finally, the firm should first use the retained earnings to finance the capital required. If the retained earnings are not sufficient, the firm should revert to selling new common stocks. If, on the other hand, retained earnings are in excess of the amount of capital expenditures needed, the firm should distribute the surplus, i.e., the residual, as dividends to shareholders. Therefore, this school of thought suggests that dividend decision is not a decision variable that affect the firm's value. In other words, this approach supports the view that dividend policy is irrelevant. This view is consistent with that of Miller and Modigliani (1961), who argue that the value of the firm is unaffected by dividend policy in a world without taxes or transaction costs. In what follows, we present Miller and Modigliani's argument.

Consider two firms that are identical in every aspect expect for their dividend payout in the current time period, i.e., at time $t = 0$. Let $\widetilde{EBIT}_i(t)$ be firm i's random future cash flow from operations at time t, $\widetilde{I}_i(t)$ be the planned investment outlay for firm i at time t, and $\widetilde{D}_i(t)$ be firm i's dividend payout in period t, for $i = 1, 2$. If we assume that both firms only differ in their dividend payout at $t = 0$, then we have the following equations:

$$\widetilde{EBIT}_1(t) = \widetilde{EBIT}_2(t) \qquad t = 0, 1, ..., \infty,$$

$$\widetilde{I}_1(t) = \widetilde{I}_2(t) \qquad t = 0, 1, ..., \infty,$$

and

$$\widetilde{D}_1(t) = \widetilde{D}_2(t) \qquad t = 1, ..., \infty,$$

but

$$\widetilde{D}_1(0) \neq \widetilde{D}_2(0).$$

In order to examine the claim that dividend policy is irrelevant, we need to answer the question of whether or not the two firms will have different value if their current dividend payout is different, but everything else is identical. To answer this question, we need a valuation model. But, first note that since both firms are identical in their expected cash flows, it is reasonable to assume that they are in the same risk class. This, in turn, implies that the required rates of return on investment projects are the same for both firms. We will also assume for simplicity that both firms are 100% equity. Now, we are ready to prove the dividend irrelevance proposition.

PROPOSITION 3. *Consider two firms that are identical in every respect except for their dividend payout in the current period. Then, in the absence of taxes and transaction costs, the value of the firm is unaffected by dividend policy.*

Let's try to prove the previous proposition. Let the setup and the notation be as mentioned above. Let the required rate of return for firm i during period t be denoted by $k(t+1)$ and defined as the sum of dividend paid in period $t+1$ plus capital gain; that is,

$$k(t+1) = \underbrace{\frac{D_i(t+1)}{P_i(t)}}_{\text{Dividend paid in \%}} + \underbrace{\frac{\widetilde{P}_i(t+1) - P_i(t)}{P_i(t)}}_{\text{capital gain in \%}}, \tag{9.15}$$

where $P_i(t)$ is firm i's share price at the end of period t, $\widetilde{P}_i(t+1)$ is firm i's share price at the end of period $(t+1)$, and $D_i(t+1)$ is firm i's dividend per share paid at the end of period t. Notice that $k(t+1)$ does not have a subscript i since both firms have the same required rate of return. Notice also that $k(t+1)$ is the rate required by each firm **during** period t **for** period

$t+1$. Let $V_i(t)$ be the market value of firm i, which is, by definition, the current number of shares outstanding, denoted by $n_i(t)$, times $P_i(t)$; that is,

$$V_i(t) = n_i(t)P_i(t).$$

Then, if we multiply (9.15) by $\frac{n_i(t)}{n_i(t)}$ and solve for $V_i(t)$, we obtain, after rearranging the terms, the equation:

$$V_i(t) = \frac{n_i(t)D_i(t+1) + n_i(t)\widetilde{P}_i(t+1)}{1+k(t+1)}. \tag{9.16}$$

Equation (9.16) states that the value of the firm is equal to the discounted sum of two cash flows, namely, the total dividend paid out $n_i(t)D_i(t+1)$ and the end of period value of the firm $n_i(t)P_i(t+1)$.

In order to prove the claim that, in a world without taxes, dividend policy does not affect the value of the firm, and thus, is irrelevant, we need to some how replace $n_i(t)D_i(t+1)$ in the numerator of (9.16). We will do this by examining the sources and uses of funds for an all-equity firm.

An all-equity firm has two major sources of funds, namely, cash generated from operations, which is $\widetilde{EBIT}_i(t+1)$ and cash generated from issuing new shares, which is the number of new shares issued, denoted by $n_i^{new}(t+1)$ times $\widetilde{P}_i(t+1)$. As for the uses of funds, there are also two main uses: dividends paid out, $n_i(t)D_i(t+1)$, and any planned cash outlays for investment, $\widetilde{I}_i(t+1)$. Since, by definition, the sources of funds must equal the uses of funds, then we have the following identity:

$$\widetilde{EBIT}_i(t+1) + n_i^{new}(t+1)\widetilde{P}_i(t+1) = n_i(t)D_i(t+1) + \widetilde{I}_i(t+1). \tag{9.17}$$

Note also that once the new shares are issued, the number of shares outstanding in period $(t+1)$ is simply the sum of the old shares and the new shares; that is,

$$n_i(t+1) = n_i(t) + n_i^{new}(t+1),$$

or equivalently,

$$n_i(t) = n_i(t+1) - n_i^{new}(t+1) \tag{9.18}$$

Substituting (9.18) in (9.16) and rearranging the terms, we get

$$V_i(t) = \frac{n_i(t)D_i(t+1) + n_i(t+1)\widetilde{P}_i(t+1) - n_i^{new}(t+1)\widetilde{P}_i(t+1)}{1+k(t+1)}. \tag{9.19}$$

Finally, express the sources and uses of funds identity in (9.17) as

$$n_i^{new}(t+1)\widetilde{P}_i(t+1) = n_i(t)D_i(t+1) + \widetilde{I}_i(t+1) - \widetilde{EBIT}_i(t+1)$$

and substitute the rearranged identity in (9.19). This yields

$$V_i(t) = \frac{n_i(t+1)\widetilde{P}_i(t+1) + \widetilde{EBIT}_i(t+1) - \widetilde{I}_i(t+1)}{1+k(t+1)}. \tag{9.20}$$

Notice that $n_i(t+1)\widetilde{P}_i(t+1)$ in (9.20) is simply the value of the firm at the end of period $t+1$. Thus,

$$V_i(t) = \frac{\widetilde{V}_i(t+1) + \widetilde{EBIT}_i(t+1) - \widetilde{I}_i(t+1)}{1+k(t+1)}. \tag{9.21}$$

Equation (9.21) clearly states that, in the absence of taxes and any transaction costs, dividend policy has not effect whatsoever on the value of the firm. This result proves the irrelevance of dividend policy in the absence of taxes.

We can also use the result from the previous proposition to prove that firms that only differ in their current dividend payout must have the same value. This is the essence of the following corollary.

COROLLARY 1. *Two firms that are identical in every respect except for their current dividend payout must have the same value.*

Consider the result in equation (9.21). First, we note that, by hypothesis,

$$\widetilde{EBIT}_1(t+1) = \widetilde{EBIT}_2(t+1)$$

and

$$\widetilde{I}_1(t+1) = \widetilde{I}_2(t+1).$$

Second, the required rate of return $k(t+1)$ is the same for both firms since they are identical. Finally, since the value of firm i at the end of period t depends on investments, dividends, and cash flows in period $t+1$ and since they are assumed to be constant, then

$$V_1(t) = V_2(t).$$

It is worth mentioning that, in addition to showing that dividend policy does not affect the value of the firm, the previous simple model also shows what does affect the value of the firm; according to the result in (9.21), the value of the firm depends only on the distribution of future generated cash flows associated with **investment decisions**. Hence, the Miller-Modigliani argument can be alternatively stated as: dividend policy has no effect on the firm's **investment decisions**. In other words, in the absence of taxes and transaction costs, the firm can decide any dividend payout it wishes without altering its investment decisions.

3.2. Arguments for Dividend Relevance. On one hand, as discussed-above, proponents of dividend irrelevance showed that dividend policy does not affect the value of the firm. On the other hand, the dividend relevance theory, which is attributed to the work of Myron Gordon (1963) and John Lintner (1962), suggests that there is a relationship between the firm's dividend policy and its market value. This theory is based on the "bird-in-hand" argument, which suggests that investors see current dividends as less risky than distant future dividend payouts. If this presumption is true, investors would prefer a bird in the hand than two in the bush! Gordon and Lintner argue that current dividend payments reduce investor uncertainty. This, in turn, causes investors to discount the firm's future earnings at a lower rate, which increases the stock and the firm's value. Conversely, if dividends are reduced or not paid, investors' uncertainty will increase. This will raise the investors' required rate of return causing them to discount the firm's future earnings at a higher rate, which will reduce the current value of the firm.

In addition to the bird-in-hand fallacy, other arguments supporting dividend relevance were set forward by researchers in the field. Empirical studies, however, failed to provide conclusive evidence in support to this intuitively appealing argument. In practice, however, as stated by Gitman (2003), financial mangers and stockholders tend to support the dividend relevance theory.

3.3. Dividend Policy in a World with Personal and Corporate Taxes. In this section we discuss the implications of considering personal and corporate taxes on the relationship between dividend policy and the firm's value. In particular, we consider Farrar and Selwyn's (1967) partial equilibrium analysis to show how dividend policy is affected by personal income tax and capital gains tax. Farrar and Selwyn assume the following:

(1) Individuals are after maximizing their after-tax income.
(2) Shareholders can either (1) own shares in an all-equity firm and **borrow** in order to provide personal leverage or (2) they can buy shares in a levered firm.
(3) The firm can either (1) pay out earnings as dividends or (2) it can retain earnings, i.e., pay no dividends, and allow shareholders to take their income in the form of capital gains.
(4) Shareholders must choose whether they seek dividends or capital gains. Dividends received by individuals are considered as part of their income, and therefore, are subject to **personal income tax**. Capital gains, however, are subject to **capital gains tax**.
(5) Individuals and firms can borrow at the same rate for simplicity.

PROPOSITION 4. *Given the previous assumptions and if capital gains tax is less than personal income tax, then shareholders who are after maximizing their after-tax income will prefer to receive their return in the form of capital gains as opposed to dividends. Hence, corporations should never pay dividends.*

Consider the choice of a representative shareholders, whose objective is to maximize after tax income. Define the following:

$\widetilde{EBIT}$ = the stochastic cash flows generated from operations.

i = the borrowing rate, which is assumed to be the same for both firms and individuals.

B_c = corporate debt.

B_p = personal debt held by the representative shareholder.

τ_c = corporate tax rate.

τ_p = personal income tax rate of the representative shareholder.

τ_g = capital gains rate for the representative shareholder.

Suppose that the firm pays out all its cash flows as **dividends**. In this case, the shareholders' income after-tax, which we will denote by $\widetilde{m}^d$ is

$$\begin{aligned} \widetilde{m}^d &= (\text{after-tax cash flow of the firm } - \text{ interest on personal debt})\,(1-\tau_p) \\ (9.22) &= \left[\left(\widetilde{EBIT} - iB_c\right)(1-\tau_c) - iB_p\right](1-\tau_p). \end{aligned}$$

Alternatively, suppose that the firm pays no dividends. In this case, we assume that all gains are taken by shareholders and taxed at the capital gains rate. The corporation can payout capital gains to shareholders instead of dividends by simply repurchasing its shares in the open market. By doing so, the firm is simply translating its cash flows into capital gains. In this case, the shareholder's after-tax income, which we will denote by $\widetilde{m}^g$, is

$$\widetilde{m}^g = \left[\left(\widetilde{EBIT} - iB_c\right)(1-\tau_c)(1-\tau_g) - iB_p\right](1-\tau_p). \tag{9.23}$$

A quick look at $\widetilde{m}^d$ in (9.22) versus $\widetilde{m}^g$ in (9.23), one can notice $\widetilde{m}^g > \widetilde{m}^d$ if $\tau_g < \tau_p$. Thus, as long as tax on capital gains is less than personal income tax, shareholders prefer to receive their returns in the form of capital gains as opposed to dividends. Hence, corporations should never pay dividends.

Brennan (1970) extended the work of Farrar and Selwyn into a more robust general equilibrium framework where shareholders are assumed to maximize their expected utility of wealth and reached a conclusion in the same spirit as that of Farrar and Selwyn (1967). In particular, Brennan argued that, for a given level of risk, the required rate of return on investment by investors is higher the higher the security's prospective dividend yield is. Brennan's rationale is that because the tax rate levied on dividends is higher than that levied on capital gains, investors should require a higher rate of return from holding securities with prospective dividend payout as opposed to capital gains.

Miller and Modigliani (1961) proved the irrelevance of dividend policy in a world without corporate taxes, transaction costs, and with symmetric information, i.e., a world where everyone is full informed about the distribution of the future uncertain cash flows. Their work is considered the foundation of the theory of dividend irrelevance. On the other hand, Farrar and Selwyn (1967) and Brennan (1970) showed that in a world with corporate and personal income taxes, it might be optimal for the firm to pay no dividends at all. Their work is considered the basis of dividend relevance theory. The academic work post the previously mentioned papers focused on providing theories that aim to explain the benefits and costs of dividend payout in hope of moving toward a theory of optimal dividends. For an excellent survey on the previous attempts, see Copeland et al (2005).